BOUNDEDNESS OF NONADDITIVE QUANTUM MEASURES

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We deal with not necessarily additive functions acting on complete orthomodular posets and taking values in Hausdorff uniform spaces, where no algebraic structure is required. As a consequence, neither pseudo-additivity, nor monotonicity are meaningful notions in this setting. Conditions ensuring their boundedness are exhibited in terms of some mild continuity properties. Such conditions are satisfied, in particular, by completely additive measures on projection lattices of von Neumann algebras. Hence, among other things, our main result provides a version in the generalized nonadditive quantum setting of the so-called boundedness principle in classical and quantum measure theory.

Keyword: nonadditive functions, orthomodular structures, von Neumann algebra, boundedness principle.

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1. Introduction

The Boundedness Principle in classical measure theory asserts that every real-valued measure acting on a σ-algebra $\mathcal{A}$ is bounded [11, Lemma 4.4]. In the more general quantum (or noncommutative) context—where $\mathcal{A}$ is actually replaced by the projection lattice $\mathcal{P}(M)$ of a von Neumann algebra M—an analogous principle fails, even when measures μ are required to be completely additive, i.e. to obey the following condition

$$\mu(\vee_{i\in I} a_i) = \sum_{i\in I} \mu(a_i) \tag{1}$$

for every set $(a_i)_{i\in I}$ of mutually orthogonal elements from $\mathcal{P}(M)$. Here, I is not necessarily countable; thus, in (1) all but countably many of the terms $\mu(a_i)$ are zero and the convergence is absolute.

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Projection structures of finite-dimensional algebras admit, in fact, many unbounded completely additive measures, as shown in [9] (see also [2, 12, 13]).

Nevertheless, the celebrated results of [10] (with [2] for simplified proofs) show that, strikingly, the mentioned failure may only occur as long as matrix algebras are taken into account. More precisely, each completely additive real-valued measure on the lattice of projections $\mathcal{P}(M)$ of a von Neumann algebra M is bounded if (and only if) M has no direct summand isomorphic to an algebra of complex matrices $M_n(\mathbb{C})$, with $n \geq 2$.

It is worth noting that projections of a von Neumann algebra form a complete lattice, not just a σ-lattice. Consequently, complete additivity for a measure is a natural requirement in the additive quantum context.

Recently there was a considerable interest on the so-called 'nonadditive framework', as finite additivity turns out to be a too much restrictive requirement in several theoretical and applied problems (see e.g. [8, 17, 21]).

The present note is aimed at providing a comprehensive approach to the problem of boundedness of not necessarily additive functions belonging to the 's-outer class'.

Loosely speaking, the membership of a function φ defined on $\mathcal{P}(M)$—and having values into a uniform space—to such a class means that $\varphi(a \vee b)$ and $\varphi(b)$ are 'close' whenever a and b in $\mathcal{P}(M)$ are orthogonal, and $\varphi(a)$ and $\varphi(0)$ are close enough (see Section 2 for a precise definition).

The interest for the 's-outer class' mainly stems from the following two facts. First, both classical finitely additive functions and various instances of nonadditive functions extensively studied in the literature (as submeasures, k-triangular functions, quasi-concave distorted measures and multisubmeasures) are s-outer functions (see [3–5] and the references therein). The s-outer class thus provides a unifying general framework. Secondly, no algebraic structure is required on target spaces of its functions. Neither pseudo-additivity, nor monotonicity are therefore meaningful notions in such a setting. Hence, arguments depending on algebraic properties inherited from target spaces—as, in particular, those in [2, 7]—cannot be employed.

Let us mention that, as underlying domains of functions, we will consider complete orthomodular posets instead of projection lattices of a von Neumann algebras. Orthomodular posets (OMPs, for short) actually generalize both projection lattices of von Neumann algebras—specifically, lattices of all projections on a Hilbert space—and Boolean algebras, since the lattice condition is dropped and the distributivity law is relaxed to the orthomodular law.

The interest for OMPs arises in connection with physical and mathematical problems. For instance, for an event structure of a quantum experiment, the lattice condition does not seem justified, whereas the distributivity law leads outside the quantum context. On the other hand, the orthomodular structures of projections in C^*-algebras, of skew projections in a Hilbert spaces and of splitting subspaces of noncomplete inner product spaces are some relevant instances of OMPs for which the lattice condition fails (see e.g. [15, 19] and the references therein).