QUASI-EXACT SOLUTIONS OF THE EQUATION FOR DESCRIPTION OF NONLINEAR WAVES IN A LIQUID WITH GAS BUBBLES

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Nonlinear partial differential equation derived by Kudryashov and Sinelshchikov for description of waves in a liquid with gas bubbles is considered. The quasi-exact solutions of this equation are found with the first, second and third order poles. Values of the residual function norm corresponding to all quasi-exact solutions are presented. It is shown that most quasi-exact solutions are transformed into exact solutions of nonlinear differential equation under some additional conditions. The found solutions can be used for description of nonlinear waves in a liquid with gas bubbles.
Keywords: Kudryashov–Sinelschikov equation, quasi-exact solution, exact solution, nonlinear ordinary differential equation.

1. Introduction

An extended equation describing long nonlinear one-dimensional waves in a liquid with gas bubbles has been derived in [1]. This equation can be presented in the form

$$u_t + \gamma u u_x + u_{xxx} - \varepsilon\,(u u_{xx})_x - \kappa u_x u_{xx} - \nu u_{xx} - \delta\,(u u_x)_x = 0, \tag{1}$$

where γ, ε, κ, ν and δ are constant parameters of the equation. Eq. (1) was obtained taking into account viscosity and heat transfer and now is often called the Kudryashov–Sinelschikov equation [2–10].

From Eq. (1) for $\varepsilon = \kappa = \nu = \delta = 0$ we can obtain the famous Korteweg–de Vries (KdV) equation [11]. We can consider Eq. (1) as the generalization of the KdV equation.

By now Eq. (1) was studied in many papers [2–10]. In [2] Ryabov has found some its exact solutions by means of the method presented in [12–15]. An application of the Lie symmetry method to Eq. (1) was considered in [3].

In [4] Randrüüt has presented phase curves corresponding to the exact solutions of Eq. (1) without dissipation and has found three types of solutions.

Eq. (1) was studied in [5–7] by using the bifurcation method for dynamic systems and the method of phase portrait analysis. Some elliptic solutions of Eq. (1) were found in [8].

In [9] the authors have shown that sech^2 type waves represent the solitary limit separating two families of periodic waves. One of them consists of the cnoidal waves that are solutions of the Korteweg–de Vries equation, while the other does not have a corresponding counterpart. In [10] the authors have shown that the traveling-wave solutions of Eq. (1) can be found from corresponding solutions of the generalized Korteweg–de Vries equation. Also a new type of periodic waves defined by this equation was constructed by combining together bounded sections of otherwise unbounded solutions of the associated generalized KdV equation.

We usually use approximate mathematical models to describe physical processes. However, after constructing mathematical model we want to have exact solutions for their description. Trying to get them we often forget that the initial equation does not correspond to the exact mathematical model. The method of quasi-exact solutions allows us to obtain an analytical description of a process which leads to an approximate mathematical model.

The aim of this work is to obtain approximate solutions of Eq. (1) using the idea of quasi-exact solutions which was proposed in [16].

Quasi-exact solutions are approximate solutions of nonlinear differential equations. They are constructed as exact solutions for some conditions of parameters. A small deviation of parameter in nonlinear differential equations leads to the quasi-exact solutions. In other words quasi-exact solution is close to exact solution for a small change of parameter of the equation. As a result, after substituting a quasi-exact solution into a nonlinear differential equation, we can obtain some small value of the residual function.

This method was applied for finding quasi-exact solutions in [16, 17] and allowed us to obtain the approximate solutions of some nonlinear differential equations. In particular, the quasi-exact solutions of the dissipative Kuramoto–Sivashinsky equation were found.

2. Quasi-exact solutions of nondissipative equation

2.1. Travelling-wave variables

Let us consider the nondissipative case ($\nu = \delta = 0$) when the two last members of Eq. (1) are insignificant. In this case Eq. (1) takes the form

$$u_t + \gamma u u_x + u_{xxx} - \varepsilon\,(u u_{xx})_x - \kappa u_x u_{xx} = 0. \tag{2}$$

Taking into account the variables

$$t = \left(\frac{\varepsilon}{\gamma}\right)^{3/2} t', \qquad x = \sqrt{\frac{\varepsilon}{\gamma}}\,x', \qquad u(x,t) = \frac{1}{\varepsilon}u'(x',t') \tag{3}$$

we obtain Eq. (2) in the form (the primes are omitted)

$$u_t + u u_x + u_{xxx} - (u u_{xx})_x - \alpha u_x u_{xx} = 0, \qquad \alpha = \frac{\kappa}{\varepsilon}. \tag{4}$$