Physica D 303 (2015) 1–17

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Signal amplification factor in stochastic resonance: An analytic non-perturbative approach

Asish Kumar Dhara

Variable Energy Cyclotron Centre, 1/AF, Bidhan Nagar, Calcutta-700064, India

HIGHLIGHTS

- We explain amplitude dependent signal amplification factor in stochastic resonance.
- The linear response theory yields amplitude independent signal amplification factor.
- The formalism proposed takes into account infinite number of perturbation terms.
- The formalism includes the contributions due to infinite number of relaxation modes.
- Only the lowest eigenfunction and Kramers' rate are needed to evaluate the response.

ARTICLE INFO

Article history:
Received 31 January 2015
Received in revised form
26 March 2015
Accepted 27 March 2015
Available online 6 April 2015
Communicated by M. Vergassola

Keywords:
Perturbation series
Fokker–Planck equation
Nonlinear response
Stochastic resonance

ABSTRACT

The response of an overdamped bistable system driven by a Gaussian white noise and perturbed by a weak monochromatic force (signal) is studied analytically. In order to get amplitude-dependent signal amplification factor a non-perturbative scheme is put forward by taking into account all the terms of a perturbation series with amplitude of the signal as an expansion parameter. An approximate analytic expression of the signal amplification factor is derived and compared with the numerical results. The contributions of infinite number of relaxation modes of the stochastic dynamics to the response are also taken into account in the final expression. The calculation of the response based on the derived expression requires only the knowledge of the first non-trivial eigenvalue and the lowest eigenfunction of the unperturbed Fokker–Planck operator.

1. Introduction

The stochastic resonance (SR) is a phenomenon where one observes non-monotonic response of a non-equilibrated non-linear system interacting with a large number of degrees of freedom and perturbed by a periodic force. The noise-induced cooperative response depends on the amplitude, A_0 and the frequency, Ω of the periodic force and the strength, D of the noisy environment in which the system is embedded. The response is usually characterized in terms of signal amplification factor which measures the amplification of the "coherent" (periodic) power of the output over the input power contained in the periodic modulation. The non-monotonic behavior of the signal amplification factor as a function of the noise strength exhibiting a maximum is considered as a characteristic of SR. The whole system exhibiting SR can then be visualized as a signal processing device to improve the amplification of a weak signal and can be used to communicate a signal to a large distance efficiently. Besides, the interest is caused by the applications relating to the enhancement of chemical reaction rates, the operation of biological motors, directed transport in Brownian ratchets, etc. Because of these reasons the noise-assisted non-equilibrium phenomena have been found to be of relevance in physics, chemistry, and the life sciences [1–3].

The framework to study this stochastic problem is the Langevin equation. Corresponding to the Langevin equation, the time evolution of the probability distribution of this stochastic process is described by the Fokker–Planck (FP) equation.

The exact solution of the Fokker–Planck Equation for this problem is not known. As the signal is weak, one analyzes the system in terms of a perturbation theory with the amplitude, A_0 of the signal as an expansion parameter. The response of this nonlinear device is thus investigated. The first term of the perturbation expansion corresponds to the linear response while the higher order terms in powers of the amplitude of the periodic force correspond to the non-linear responses.

It has been possible to obtain approximate analytical expressions [1,4,5] of the linear response. The non-linear SR responses

E-mail address: akd@vecc.gov.in.

http://dx.doi.org/10.1016/j.physd.2015.03.008
0167-2789/

(signal amplification factor) have been calculated [4] by solving the Langevin equation numerically without having a recourse to the perturbation theory.

Linear response approximation (considering only the first term of the series) provides the signal amplification factor independent of the amplitude of the input periodic signal while the numerical results show that the response (signal amplification factor) does depend on the amplitude. This suggests that in order to estimate true response (amplitude-dependent) one should calculate the non-linear response i.e., higher order terms in the perturbation series. Recently it has been possible to obtain an approximate analytic expression [6] for the leading-order non-linear response. Complexity grows for the calculation of the next to leading order (NLO) and the next-to next leading order (NNLO) non-linear response analytically. It is observed that the successive terms (NLO, NNLO) although appear with alternate signs but the coefficients, $fac_j(\Omega, D)$ of the derived expression of the response, $\sum_j \left(\frac{A_0}{D}\right)^j fac_j(\Omega, D)$ are not sufficiently small to annul the large enhancement due to the factor $\left(\frac{A_0}{D}\right)^j$. How many terms one would retain in the perturbation series to have a reasonable finite response is not known. Thus the usual method of perturbation approach by truncating the series fails in this case. One wonders while considering the full perturbation series to estimate the non-linear response even with moderately low amplitude of the signal these oscillations would be violent for higher and higher order terms and how these terms cancel in a subtle way to arrive at a finite response [4,7]. This difficulty motivates us to put forward a new method to calculate the amplitude-dependent signal amplification factor in a non-perturbative way taking into account the infinite number of terms of the perturbation series.

As has been stated before, the difficulty to handle the higher order terms of the series grows enormously. One is therefore forced to look for an approximate approach to manage the terms efficiently as far as possible. In this present paper we develop a scheme to derive the non-perturbative response analytically taking into account the infinite number of terms of the perturbation series. The stochastic dynamics depends on the infinite number of relaxation modes for the problem considered in this manuscript. These modes are characterized by the eigenvalues of the un-perturbed Fokker–Planck operator. The response is calculated in this manuscript taking into account

(i) infinite number of terms of the perturbation series and
(ii) the contributions arising from infinite number of relaxation modes.

It is thus easily envisaged that the task is highly non-trivial. To begin with, in order to carry out this program we largely appeal to simplicity by restricting a suitable domain of the parameter regime (A_0, Ω, D). Further, throughout the derivation we always ignore quantities which are of higher order smallness compared to the accounted terms.

The paper is organized as follows. We state the problem briefly in Section 2. The interaction of the input monochromatic signal with the un-perturbed stochastic system generates harmonics of the signal frequency at the output. This is exhibited in Eq. (2.9). In this manuscript we calculate the signal amplification factor of a monochromatic periodic signal which is considered as a quantifier of stochastic resonance. It is defined as the ratio of the signal power at the output to that at the input and it is given by Eq. (2.14). As the power contained in the input monochromatic signal is $\frac{A_0^2}{2}$, hence $\bar{C}_{coh}(\Omega, D, A_0)$ in Eq. (2.13) is a measure of the power at the output. This implies that $\langle\phi_0^\dagger|x|C_1(\mu = 0)\rangle$ (with $\phi_0^\dagger(x) = 1$ and $C_1(x; \mu = 0)$ being the magnitude of the fundamental present in the asymptotic probability distribution, Eq. (2.9)) in Eq. (2.13) is the effective amplitude of the signal at the output. Since we are taking into account infinite number of terms of the perturbation series, the quantity, $C_1(x; \mu = 0)$ is expressed as a series with its different orders in Eq. (3.13) where the amplitude A_0 is an expansion parameter. We calculate the amplitude through spectral decomposition of $C_1(x; \mu = 0)$ with respect to the complete set of eigenfunctions of the un-perturbed Fokker–Planck operator. The spectral decomposition of different orders of C_1 shows that the spectral component of a given order of C_1 is related to the various spectral components of lower order of C_1 and the third harmonic component, C_3. This is obtained as the infinite hierarchial system of equations in Eq. (3.14). This set of equations explicitly demonstrates that in all orders of perturbation one has to take into account the contributions arising from all the infinite number of relaxation modes of the stochastic process. We solve this hierarchy of the spectral components to calculate the amplitude at the output in Eq. (3.16). In order to solve this hierarchy analytically we restrict the domain of the parameters, (A_0, Ω, D) of this problem where Eqs. (3.17)–(3.19) hold. This would help us to write this hierarchy in a compact form given in Eq. (3.24). These are presented in Section 3. We next solve this hierarchy in two steps. It is seen that this infinite set of hierarchy involves the spectral components of C_1 and C_3. In Section 4 we solve this hierarchy ignoring the contribution due to third harmonic and obtain the solution of the spectral components of C_1 and the corresponding resonance amplitude in Eqs. (4.8) and (4.19) respectively. We call them as ${}^{(1)}\langle\phi_m^\dagger|C_1^{(2k+1)}(\mu = 0)\rangle$ and ${}^{(1)}\langle\phi_0^\dagger|x|C_1(\mu = 0)\rangle$ for convenience. In Section 5 we solve this hierarchy including the contribution due to third harmonic and obtain the resonance amplitude, Eq. (5.14) as a series, where each term of the series can be evaluated with the knowledge of the solution obtained in Eq. (4.8). To demonstrate the validity of the solution of the resonance amplitude, Eq. (5.14) we evaluate two typical string of infinite number of perturbation terms in Sections 5.1 and 5.2 respectively. Using Eqs. (2.13)–(2.14) we evaluate approximate signal amplification factor for the two typical amplitudes of the input signal, namely $A_0 = 0.1$ and $A_0 = 0.2$, and compare the analytical results with the numerical values available in the literature in Figs. 3 and 4. For both the cases the amplitude-dependent response show a non-monotonic behavior exhibiting a maximum (the characteristic feature of SR) as in the numerical observation. As the truncation of the usual perturbation series is avoided, the finite amplitude-dependent signal amplification factor is achieved in this non-perturbative approach. Finally some concluding remarks are added in Section 6.

As infinite number of terms in the perturbation series and the contributions from infinite number of relaxation modes are considered in this approach, the derivation becomes non-trivial and the expression of the response amplitude is somewhat involved. Nevertheless the evaluation of the non-linear response amplitude or the corresponding signal amplification factor needs only the knowledge of the lowest eigenfunction and the first nontrivial eigenvalue of the un-perturbed FP operator. Necessary steps to arrive at the final result are proved in the appendices.

2. Statement of the problem

The Langevin equation describing the overdamped Brownian motion of a particle in a bistable potential $V(x)$, driven by a Gaussian white noise and perturbed by a weak monochromatic force $A_0 \cos(\Omega t)$ (the input signal), is given by

$$\dot{x} = -V'(x) + A_0 \cos(\Omega t) + \Gamma(t). \tag{2.1}$$

The bistable potential $V(x)$ used in Eq. (2.1) is

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4, \tag{2.2}$$