COMPACTON-LIKE SOLUTIONS OF THE HYDRODYNAMIC SYSTEM DESCRIBING RELAXING MEDIA

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We show the existence of a compacton-like solutions within the relaxing hydrodynamic-type model and perform numerical study of attracting features of these solutions.

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1. Introduction

In this paper there are studied solutions to evolutionary equations, describing wave patterns with compact support. Different kinds of wave patterns play the key rôle in natural processes. They occur in nonlinear transport phenomena (see [1] and references therein), serve as channels of information transfer in animate systems [2], and very often assure stability of some dynamical processes [3, 4].

One of the most advanced mathematical theory dealing with the formation of wave patterns and evolution is the soliton theory [5]. The origin of this theory goes back to Scott Russell's description of the solitary wave movement on the surface of channel filled with water. It was the ability of the wave to move quite a long distance without any change of shape which stroke the imagination of the first chronicler of this phenomenon. In 1895 Korteveg and de Vries proposed their famous equation

$$u_t + \beta\, u\, u_x + u_{xxx} = 0, \tag{1}$$

describing evolution of long waves on shallow water. They also obtained an analytical solution to this equation, corresponding to the solitary wave:

$$u = \frac{12\, a^2}{\beta}\, \mathrm{sech}^2\big[a(x - 4\, a^2\, t)\big]. \tag{2}$$

Both the already mentioned report by Scott Russell as well as the model suggested to explain his observation did not find a proper impact till the middle of 60-ies of the XX century when there was established a number of outstanding features of Eq. (1) finally recognized as the consequences of its complete integrability [5].

In recent years there has been discovered another type of solitary waves referred to as *compactons* [6]. These solutions inherit main features of solitons, but differ from them in one point: their supports are compact.

A big progress is actually observed in studying compactons and their properties, yet most papers dealing with this subject are concerned with compactons as solutions to either completely integrable equations, or those which produce a completely integrable ones when reduced to a subset of traveling wave (TW) solutions [7–9].

In this paper the compacton-like solutions to the hydrodynamic-type model taking account of the effects of temporal nonlocality are studied. Being of dissipative type, this model is obviously non-Hamiltonian. As a consequence, compactons exist merely for selected values of the parameters. In spite of such restriction, the existence of these solutions in significant for they appear in the presence of relaxing effects and rather cannot exist in any local hydrodynamic model. Besides, the compacton-like solutions manifest attractive features and can be treated as some universal mechanism of the energy transfer in media with internal structure, leading to the given type of the hydrodynamic-type modeling system.

The structure of the paper is as follows. In Section 2 we give a geometric insight into the soliton and compacton TW solutions, revealing the mechanism of appearance of generalized solutions with compact supports. In Section 3 we introduce the modeling system and show that compacton-like solutions do exist among the set of TW solutions. In Section 4 we perform numerical investigations of the modeling system based on the Godunov method and show that the compacton-like solutions manifest attractive features.

2. Solitons and compactons from the geometric viewpoint

Let us discuss how the solitary wave solution to (1) can be obtained. Since the function $u(\cdot)$ in formula (2) depends on the specific combination of the independent variables, we can use for this purpose the ansatz $u(t, x) = U(\xi)$, with $\xi = x - V t$. Inserting this ansatz into Eq. (1) we get, after one integration, the following system of ODEs:

$$\begin{aligned} \dot{U}(\xi) &= -W(\xi), \\ \dot{W}(\xi) &= \frac{\beta}{2} U(\xi) \left(U(\xi) - \frac{2V}{\beta} \right). \end{aligned} \tag{3}$$

The system (3) is a Hamiltonian system described by the Hamilton function

$$H = \frac{1}{2} \left(W^2 + \frac{\beta}{3} U^3 - V U^2 \right). \tag{4}$$

So every solutions of (3) can be identified with some level curve $H = C$. As already mentioned the solution (2) corresponds to the value $C = 0$ and is represented by the homoclinic trajectory shown in Fig. 1 (the only trajectory in the right half-plane going through the origin). Since the origin is an equilibrium point of (3) and penetration of the homoclinic loop takes infinite "time", then the beginning of this