ON GRADED ASSOCIATIVE ALGEBRAS

ANTONIO J. CALDERÓN MARTÍN*

Departamento de Matemáticas,
Universidad de Cádiz, 11510 Puerto Real, Cádiz, Spain
(e-mail: ajesus.calderon@uca.es)

(*Received April 26, 2011 – Revised October 24, 2011*)

Consider $\mathfrak{A}$ an associative algebra of arbitrary dimension and over an arbitrary base field $\mathbb{K}$, graded by means of an abelian group G. We show that $\mathfrak{A}$ is of the form $\mathfrak{A} = U + \sum_j I_j$ with U a linear subspace of $\mathfrak{A}_1$ and any I_j a well described graded ideal of $\mathfrak{A}$, satisfying $I_j I_k = 0$ if $j \neq k$. Under certain conditions, the simplicity of $\mathfrak{A}$ is characterized and it is shown that $\mathfrak{A}$ is the direct sum of the family of its minimal graded ideals, each one being a simple graded associative algebra.

Keywords: graded algebra, associative algebra, structure theory.

1. Introduction and previous definitions

The interest in gradings on different classes of algebras has been remarkable in the last years specially motivated by their application in physics, see [2, 7, 12, 14, 1, 11, 19, 24–26]. In particular, graded associative algebras are interesting not only by themselves but also because we can derive from them many examples of graded Lie algebras, which play an important role in the theory of strings, color supergravity, Walsh functions or electroweak interactions [8, 9, 18, 21, 23, 30]. In the present paper we study arbitrary graded associative algebras (not necessarily simple or finite-dimensional), and over an arbitrary field $\mathbb{K}$ by focussing on their structure. Hence throughout this paper we consider associative algebras $\mathfrak{A}$ of arbitrary dimension and over an arbitrary base field $\mathbb{K}$.

DEFINITION 1.1. Let $\mathfrak{A}$ be an associative algebra and G an abelian group. It is said that $\mathfrak{A}$ is a *graded associative algebra*, by means of G, if

$$\mathfrak{A} = \bigoplus_{g \in G} \mathfrak{A}_g,$$

where any $\mathfrak{A}_g$ is a linear subspace satisfying $\mathfrak{A}_g \mathfrak{A}_h \subset \mathfrak{A}_{gh}$ (denoting by juxtaposition the product in G), for any $h \in G$.

*Supported by the PCI of the UCA 'Teoría de Lie y Teoría de Espacios de Banach', by the PAI with project numbers FQM298, FQM2467, FQM3737 and by the project of the Spanish Ministerio de Educación y Ciencia MTM2007-60333.

We call the *support* of the grading the set

$$\Sigma := \{g \in G \setminus 1 : \mathfrak{A}_g \neq 0\}.$$

The support of the grading is called *symmetric* if $g \in \Sigma$ implies $g^{-1} \in \Sigma$.

A *graded ideal* I of $\mathfrak{A}$ is an ideal which splits as $I = \bigoplus_{g\in G} I_g$ with any $I_g = I \cap \mathfrak{A}_g$. A graded associative algebra $\mathfrak{A}$ will be called *simple* if its product is nonzero and its only graded ideals are $\{0\}$ and $\mathfrak{A}$.

In Section 2 we develop connections techniques on the support of the grading so as to show that $\mathfrak{A}$ is of the form $\mathfrak{A} = U + \sum_j I_j$ with U a subspace of $\mathfrak{A}_1$ and any I_j a well-described graded ideal of $\mathfrak{A}$, satisfying $I_j I_k = 0$ if $j \neq k$. In Section 3, and under certain conditions, the simplicity of $\mathfrak{A}$ is characterized and it is shown that $\mathfrak{A}$ is the direct sum of the family of its minimal graded ideals, each one being a simple graded associative algebra.

2. Decompositions

From now on, $\mathfrak{A}$ denotes a graded associative algebra over the base field $\mathbb{K}$ with a symmetric support Σ and

$$\mathfrak{A} = \bigoplus_{g\in G} \mathfrak{A}_g = \mathfrak{A}_1 \oplus \Big(\bigoplus_{g\in\Sigma} \mathfrak{A}_g\Big)$$

the corresponding grading. We begin by developing connection techniques in this framework.

DEFINITION 2.1. Let g and g' be two elements in Σ. We shall say that g is *connected* to g' if there exist $g_1, g_2 \dots, g_n \in \Sigma$ such that

1. $g_1 = g$.
2. $\{g_1, g_1g_2, g_1g_2g_3, \dots, g_1g_2g_3\cdots g_{n-2}g_{n-1}\} \subset \Sigma$.
3. $g_1g_2g_3\cdots g_{n-1}g_n \in \{g', (g')^{-1}\}$.

We shall also say that $\{g_1, \dots, g_n\}$ is a *connection* from g to g'.

Observe that $\{g\}$ is a connection from g to itself and to g^{-1} and so g is connected to g and to g^{-1}.

The next result shows that the connection relation is an equivalence.

PROPOSITION 2.1. *The relation $\sim$ in Σ defined by $g \sim g'$ if and only if g is connected to g' is an equivalence.*

Proof: $\{g\}$ is a connection from g to itself and therefore $g \sim g$.

Let us see the symmetric character of $\sim$. If $g \sim g'$, there exists a connection

$$\{g_1, g_2, g_3, \dots, g_{n-1}, g_n\} \subset \Sigma$$

from g to g'. Then $g_1 = g$,

$$\{g_1, g_1g_2, \dots, g_1g_2\cdots g_{n-1}\} \subset \Sigma, \tag{1}$$