FINSLEROID-FINSLER PARALLELISM. COSMOLOGICAL ASPECTS

G. S. ASANOV

Division of Theoretical Physics, Moscow State University,
119992 Moscow, Russia
(e-mail: asanov@newmail.ru)

(Received November 15, 2006)

The lucky (and very unique) position of the Finsleroid-Finsler geometry is the occurrence of a simple algebraic representation for the scalar product, and hence for the angle, of two vectors. In the present paper it is shown that the Finsleroid-type parallel transportation of vectors retains the scalar product and angle unchanged, in the Landsberg-type case. The two-vector extension of the Finsleroid metric tensor is proposed. The Total Category of Parallelism is obtained, in which all the distant-parallelism concepts are meaningful as well as representable in an explicit and simple way. The conclusions made can be reformulated to apply in the relativistic pseudo-Finsleroid framework. Respective gravitational field equations and cosmological metric are discussed.

Keywords: Finsler geometry, metric spaces, angle, scalar product, parallelism, relativity.

1. Introduction and synopsis

The principal position of the Riemannian geometry is the phenomenon that the angle between vectors does not change under the parallel transportation of the vectors. The theory of connection in the Riemannian geometry is developed, and taught to students, subject to this observation. Can the phenomenon be transgressed from the Riemannian geometry to the Finsler geometry? No transparent and constructive answer is suggested by the content of current literature devoted to Finsler spaces (see the books [1–3]). This notwithstanding, quite certain positive answer proves to be a truth in the domain of the Finsleroid-Finsler geometry outlined in [4–9]. The answer is gained in the following succession of steps. Firstly, we use the Finsleroid-produced scalar product $\langle y_1, y_2\rangle_x$ obtained by studying the equations of geodesics in tangent spaces. Secondly, we define the parallel displacement (1.1) of vector y^i with the respective spray-induced coefficients $\bar{G}^i{}_m$. Thirdly, we apply the Landsberg-case spray coefficients. Lastly, we verify by straightforward calculations (which are short and easy) that such a procedure does not change the value of $\langle y_1, y_2\rangle_x$ and, hence, the Finsleroid angle.

Therefore, fixing the Landsberg case, we are entitled to conclude that the Finsleroid approach proves to overcome the vague opinion that in the Finsler geometry scientists may be "in principle equipped with only a family of Minkowski norms", so that "yardsticks are assigned, but protractors are not". They can be equipped also with a convenient family of the two-vector products $\langle y_1, y_2\rangle_x$, thereby with "protractors"!

This circumstance thrusts forth new questions fundamental to the very realm of the nowaday Finsler geometry: should we consider that geometry "old-fashioned" from the new advantageous standpoint that is proposed by the Finsleroid-induced geometry? The vantage-ground answer is "No" in many principle aspects, particularly the concepts of the Finslerian metric function and metric tensor, the Cartan tensor, the geodesics and spray coefficients, the significance and geometry of indicatrix, the nonlinear covariant derivative, the connection and curvature on the tangent bundle, the flag curvature, etc., are keeping fine. Simultaneously, the answer is decisively "Yes" in numerous new respects, including the occurrence of the scalar product $\langle y_1, y_2\rangle_x$ between two vectors from which many new categories of the cardinal geometrical nature proper are stemming up.

Among such categories, "the transportation preserving angle between two vectors" is notable, and can be consistently and explicitly tractable. Indeed, we may define the *Finsleroid-Finsler covariant differential* δy of a vector y along (horizontal) dx in the natural way

$$\delta y^i \ := dy^i + \bar{G}^i{}_k(x,y)dx^k, \tag{1.1}$$

where $\bar{G}^i{}_k = \frac{1}{2}\partial G^i/\partial y^k$ and $G^i \ := \gamma^i{}_{mn}y^m y^n$ are the respective spray coefficients, with $\gamma^i{}_{mn}$ standing for the Finslerian Christoffel symbols constructed from the Finsleroid-Finsler metric function K. The vector y is said to undergo the *parallel transportation along* dx if

$$\delta y = 0. \tag{1.2}$$

Then analytically the condition for the scalar product $\langle y_1, y_2\rangle_x$ to be unchanged under such a transportation of vectors y_1, y_2 reads

$$\frac{\partial\langle y_1, y_2\rangle_x}{\partial x^k} - \bar{G}^n{}_k(x, y_1)\frac{\partial\langle y_1, y_2\rangle_x}{\partial y_1^n} - \bar{G}^n{}_k(x, y_2)\frac{\partial\langle y_1, y_2\rangle_x}{\partial y_2^n} = 0. \tag{1.3}$$

We claim the following.

PARALLEL TRANSPORTATION THEOREM. *In the Landsberg case of the Finsleroid--Finsler space, the condition* (1.3) *holds fine.*

The proof is arrived at after direct calculations which are not lengthy, as will be demonstrated in Appendix A. Thus, both the scalar product (angle) of pair of vectors as well the parallel transportation retaining the product (angle) can nicely be transgressed from the Riemannian geometry to the Finsleroid-Finsler geometry in a simple analytical way. *This parallelism in the Finsleroid domain comes to play replacing the Levi-Civita parallelism functioned conventionally in the Riemannian geometry.*

In Finsler geometry, we have two concepts of vector length. Namely, we can use the Finslerian metric function $F = \sqrt{g_{ij}(x,y)y^i y^j}$ to assign the *absolute length*

$$\|y\|_x = \sqrt{g_{ij}(x,y)y^i y^j} \tag{1.4}$$