THE EXOCENTER OF A GENERALIZED EFFECT ALGEBRA

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Elements of the exocenter of a generalized effect algebra (GEA) correspond to decompositions of the GEA as a direct sum and thus the exocenter is a generalization to GEAs of the center of an effect algebra. The exocenter of a GEA is shown to be a boolean algebra, and the notion of a hull mapping for an effect algebra is generalized to a hull system for a GEA. We study Dedekind orthocompleteness of GEAs and extend to GEAs the notion of a centrally orthocomplete effect algebra.

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1. Introduction

Our purpose in this article is to define and study extensions to generalized effect algebras of the notions of the center [7], orthocompleteness [13], central orthocompleteness [4], hull mappings [5], and the central cover [5] for an effect algebra.

Effect algebras were originally introduced as algebraic bases for the theory of quantum measurement, particularly measurements that involve fuzziness or unsharpness [3]. Special kinds of effect algebras include orthoalgebras, MV-algebras, Heyting MV-algebras, orthomodular posets, orthomodular lattices, and boolean algebras. An account of the axiomatic approach to quantum mechanics based on orthomodular lattices or orthomodular posets with suitably rich sets of probability measures, developed circa 1955–1991, can be found in [23]. Later work, based on

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effect algebras and the closely related D-posets, is summarized in [2]. Increased interest in generalized effect algebras (GEAs) is partially due to the discovery that various systems of (possibly) unbounded symmetric operators on a Hilbert space, in particular, operators that represent quantum observables and states, can be organized into GEAs [19, 20, 22, 25].

The notion of a generalized boolean algebra—roughly speaking, a "boolean algebra without a unit"—goes back to the pioneering work of M.H. Stone in the 1930s [27]. Whereas a boolean algebra is categorically equivalent to an idempotent ring with unit, a generalized boolean algebra is categorically equivalent to an idempotent ring that may fail to have a unit. In 1968, M.F. Janowitz extended Stone's earlier work by defining and studying generalized orthomodular lattices—again roughly "orthomodular lattices without units"—and he showed that such a lattice can be realized as a prime ideal in an orthomodular lattice [10]. Results extending those of Stone and Janowitz were obtained by J. Hedlíková and S. Pulmannová for generalized orthoalgebras [9] and by A. Mayet-Ippolito for generalized orthomodular posets [17].

Extension of the work of Stone, Janowitz, et al. leading to a theory of generalized effect algebras has been conducted by a number of authors, including F. Kôpka and F. Chovanec [16] (in the context of D-posets), D.J. Foulis and M.K. Bennett [3] (in terms of positive cones in partially ordered abelian groups), G. Kalmbach and Z. Riečanová [15] (for abelian RI-posets and abelian RI-semigroups), and A. Wilce [28] as well as J. Hedlíková and S. Pulmannová [9] (in connection with cancellative positive partial abelian semigroups). In [24, §2], Riečanová observed that D-posets, cones, abelian RI-posets, abelian RI-semigroups, and cancellative positive partial abelian semigroups all may be regarded as *generalized effect algebras* in the sense of Definition 2.1 below.

2. Generalized effect algebras

In this section we review some of the definitions pertaining to and some of the elementary properties of generalized effect algebras (GEAs)—see [2, 9, 11, 12, 17, 18, 21, 24, 28] for more details. We begin by recalling the definition of a GEA.

DEFINITION 2.1 [24, Definition 2.1]. A *generalized effect algebra* (GEA) is a partial algebra $(E;\oplus,0)$ with a partially defined binary operation $\oplus$, called the *orthosummation*, and with a constant $0\in E$, called the *zero*, such that conditions (GEA1)–(GEA5) below are satisfied for all $e, f, d\in E$:
(GEA1) $e\oplus f=f\oplus e$ (*commutativity*);
(GEA2) $(e\oplus f)\oplus d=e\oplus(f\oplus d)$ (*associativity*);
(GEA3) $e\oplus 0=e$ (*neutral element*);
(GEA4) $e\oplus f=e\oplus d\Rightarrow f=d$ (*cancellation*);
(GEA5) $e\oplus f=0\Rightarrow e=0=f$ (*positivity*).

Equalities in GEA1 and GEA2 are understood in the sense that, if one side is defined, so is the other, and then the equality holds. As observed in [12, §1], a GEA