Physica D 289 (2014) 18–31

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

The Kuramoto model of coupled oscillators with a bi-harmonic coupling function

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HIGHLIGHTS

- The Kuramoto model with a bi-harmonic coupling function was investigated.
- We develop a method for an analytic solution of self-consistent equations.
- We observed a multi-branch locking with a multiplicity of coherent states.
- Multi-branch synchronous states coexist with neutrally stable asynchronous regime.
- We show that the asynchronous state has a finite life time for finite ensembles.

ARTICLE INFO

Article history:
Received 29 April 2014
Received in revised form
30 August 2014
Accepted 3 September 2014
Available online 16 September 2014
Communicated by S. Coombes

Keywords:
Kuramoto model
Bi-harmonic coupling function
Multi-branch entrainment
Synchronization

ABSTRACT

We study synchronization in a Kuramoto model of globally coupled phase oscillators with a bi-harmonic coupling function, in the thermodynamic limit of large populations. We develop a method for an analytic solution of self-consistent equations describing uniformly rotating complex order parameters, both for single-branch (one possible state of locked oscillators) and multi-branch (two possible values of locked phases) entrainment. We show that synchronous states coexist with the neutrally linearly stable asynchronous regime. The latter has a finite life time for finite ensembles, this time grows with the ensemble size as a power law.

1. Introduction

Large systems of coupled nonidentical oscillators are of general interest in various branches of science. They describe Josephson junction circuits [1–3], electrochemical [4] and spin-torque [5,6] oscillators, as well as variety of interdisciplinary applications including pedestrian induced oscillations of footbridges [7], applauding persons [8], and others. Similar models are also used in biology, for example in studying of neural ensembles dynamics [9,10] and systems describing circadian clocks in mammals [11,12]. In many cases the analysis of large ensembles consisting of heterogeneous oscillators can be successfully performed in the phase approximation [13,14]. Indeed, if the interaction between the elements is weak, the amplitudes are enslaved, and the dynamics of self-sustained oscillators can be effectively described by a relatively simple system of coupled phase equations. The special case of a globally coupled network of phase oscillators (so-called Kuramoto model [13,15]) attracted a lot of attention [16] and has been established as a paradigmatic model describing transitions from incoherent to synchronous states in the ensembles of coupled oscillators.

Quite a complete analysis of the Kuramoto model can be performed in the case of a harmonic sin-coupling function [13,17,18], although even here non-trivial scenarios of transition to synchrony have been reported [19]. Less studied is the case of more general coupling functions, containing many harmonics. Here we perform a systematic study of the synchronous regimes for a bi-harmonic coupling function (see [20] for a short presentation of these results which have been later confirmed in [21]). We introduce the model and discuss previous findings in Section 2. Then in Section 3 we give a general solution of the self-consistent equations describing rotating-wave synchronous solutions. In Section 4 we give a detailed analysis of the simplest symmetric

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http://dx.doi.org/10.1016/j.physd.2014.09.002
0167-2789/

case (no phase shifts in the coupling), while a general situation is illustrated in Section 5. In conclusion, we summarize the results and outline open questions. In this paper we focus on the deterministic oscillator dynamics, the case of noisy oscillators will be considered elsewhere [22].

2. Kuramoto model and bi-harmonic coupling

The general Kuramoto model is formulated as a system of differential equations for the phases φ_k of N oscillators:

$$\dot{\varphi}_k = \omega_k + \frac{1}{N}\sum_{n=1}^{N}\Gamma(\varphi_n - \varphi_k), \quad k = 1, \dots, N. \tag{1}$$

All the oscillators are identical, except for diversity of the natural frequencies ω_k, distributed according to a certain distribution function $g(\omega)$. The level of coherence in the network of phase oscillators can be described by order parameters R_n, defined as:

$$R_n e^{i\Theta_n} = \frac{1}{N}\sum_{k=1}^{N} e^{in\varphi_k}, \quad n \in \mathbb{N}. \tag{2}$$

The state with $R_n = 0$ for all n corresponds to a purely incoherent dynamics (uniform distribution of the phases), while non-zero values of at least some order parameters indicate for certain synchrony in the ensemble. In the case of pure sinusoidal coupling, $\Gamma(x) = \varepsilon\sin(x + \alpha)$, the original analysis by Kuramoto [15,13] and its subsequent extensions [23–25,17,18] revealed a clear picture of a transition from an asynchronous state to coherence in the thermodynamical limit $N \to \infty$. It was shown that above certain critical value of the coupling ($\varepsilon > \varepsilon_c$), the system undergoes a transition from disordered behavior to synchronous collective motion via a supercritical bifurcation with the main order parameter obeying $R_1 \sim (\varepsilon - \varepsilon_c)^{\frac{1}{2}}$.

The situation is much less trivial for more general coupling functions Γ. The presence of higher harmonics in coupling function [26,24,25,27] may change scaling of the order parameter to linear law $R_1 \sim \varepsilon - \varepsilon_c$. Moreover, as has been already mentioned in an early paper by Winfree [28] and in subsequent numerical studies by Daido in [29,30], sufficiently strong higher modes in the coupling function Γ may cause a so-called multibranch entrainment, in which a huge number of stable or multistable phase-locked states exist. In certain cases the interplay between synchronizing action of one coupling mode and repelling force from another one can be a reason for an oscillatory behavior of macroscopic order parameters [31].

This paper is devoted to a systematic study of the Kuramoto model in the case of a general bi-harmonic coupling function

$$\Gamma(x) = \varepsilon\sin(x - \beta_1) + \gamma\sin(2x - \beta_2) \tag{3}$$

in the thermodynamic limit $N \to \infty$. In Section 3 we formulate an analytic self-consistent approach [15,13,32] which allows us to calculate stationary or uniformly rotating order parameters $R_{1,2}$ (including all possible multi-branch entrainment states) depending on the parameters of the bi-harmonic coupling function Γ. Based on the self-consistent method, we present in Section 4 a complete diagram of uniformly rotating states with constant order parameters, for a special case of symmetric coupling function Γ ($\beta_{1,2} = 0$). Surprisingly, (i) synchronous solutions appear *prior* to the stability threshold of incoherent state; (ii) these regimes have order parameters that can take values anywhere in the range $(0, R_{max}]$ for some $R_{max} < 1$; (iii) there is a huge multiplicity of these states for fixed coupling parameters (multi-branch entrainment) which can also appear for *relatively weak* second mode (when parameter γ is small compared to absolute value of ε) in the coupling. Here we also illustrate the multiplicity of solutions, and, combining the self-consistent approach and a perturbative analysis, we derive the scaling laws of $R_{1,2}(\varepsilon, \gamma)$ near the transition points where coherent state appears.

For a general case of non-zero phase shifts $\beta_{1,2}$, consideration of the self-consistent equations becomes rather tedious due to a large number of parameters involved. We restrict our attention in Section 5 to several examples with multibranch entrainment and already mentioned oscillatory states [31].

Before proceeding with the analysis, we mention three examples of realistic physical systems where the second harmonics term in the coupling function is strong or even dominating. The first example is the classical Huygens' setup with pendulum clocks suspended on a common beam (common platform). The horizontal displacement of the beam leads to the first harmonics coupling $\sim \varepsilon$, while the vertical mode produces the second harmonics term $\sim \gamma$ [33]. We give a derivation of the phase equations for the case where both horizontal and vertical displacements of the platform are present, in Appendix, where Eq. (32) is in fact the Kuramoto model with bi-harmonic coupling. Another example are recently experimentally realized φ-Josephson junctions [34], where the dynamics of a single junction in the array is governed by a double-well energy potential. Therefore one can expect strong effects caused by the second harmonics in the interaction. The third example are experiments with globally coupled electrochemical oscillators [35,36], where a pronounced second harmonics has been observed in the coupling function inferred from the experimental data.

3. Self-consistent equations and their solution

We start our analysis with reformulation of Eq. (1) for the bi-harmonic coupling as

$$\dot{\varphi}_k = \omega_k + \varepsilon\,\mathrm{Im}\left[e^{-i\beta_1 - i\varphi_k}\frac{1}{N}\sum_n e^{i\varphi_n}\right] + \gamma\,\mathrm{Im}\left[e^{-i\beta_2 - i2\varphi_k}\frac{1}{N}\sum_n e^{i2\varphi_n}\right].$$

In the thermodynamical limit, using the two relevant order parameters $R_{1,2}e^{i\Theta_{1,2}}$, defined according to (2), we obtain:

$$\dot{\varphi} = \omega + \varepsilon R_1\sin(\Theta_1 - \varphi - \beta_1) + \gamma R_2\sin(\Theta_2 - 2\varphi - \beta_2). \tag{4}$$

We assume the natural frequencies ω to be distributed according to a symmetric unimodal density $g(\omega)$. Furthermore, due to rotational invariance of the problem, the mean frequency can be set to zero by virtue of a transformation into a rotating reference frame. In the thermodynamic limit the complex order parameters $R_m e^{i\Theta_m}$ can be represented using the conditional distribution function $\rho(\varphi|\omega)$:

$$R_m e^{i\Theta_m} = \iint d\varphi\, d\omega\; g(\omega)\rho(\varphi|\omega)e^{im\varphi}, \quad m = 1, 2. \tag{5}$$

Below we consider only the states of uniformly rotating order parameters. Let us perform the following transformation of variables to the rotating (with some frequency Ω) reference frame:

$$\Theta_1 = \Omega t + \theta_1; \qquad \Theta_2 = 2\Omega t + \theta_2; \\ \varphi = \Omega t + \theta_1 - \beta_1 + \psi, \tag{6}$$

where θ_1 and θ_2 are constants. Then Eq. (4) changes as follows:

$$\dot{\psi} = \omega - \Omega + \varepsilon R_1\sin(-\psi) + \gamma R_2\sin(\theta_2 - 2\theta_1 + 2\beta_1 - \beta_2 - 2\psi). \tag{7}$$