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Exact and approximate solutions for optical solitary waves in nematic liquid crystals

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HIGHLIGHTS

- Exact solutions for solitary waves in liquid crystals found.
- Bistability of solitary waves in liquid crystals found.
- Reasons for stability of solitary waves in liquid crystals identified.
- Variational approximations for solitary waves found and their accuracy determined.
- The existence of minimum power solitary waves investigated.

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ABSTRACT

The equations governing optical solitary waves in nonlinear nematic liquid crystals are investigated in both $(1+1)$ and $(2+1)$ dimensions. An isolated exact solitary wave solution is found in $(1+1)$ dimensions and an isolated, exact, radially symmetric solitary wave solution is found in $(2+1)$ dimensions. These exact solutions are used to elucidate what is meant by a nematic liquid crystal to have a nonlocal response and the full role of this nonlocal response in the stability of $(2+1)$ dimensional solitary waves. General, approximate solitary wave solutions in $(1+1)$ and $(2+1)$ dimensions are found using variational methods and they are found to be in excellent agreement with the full numerical solutions. These variational solutions predict that a minimum optical power is required for a solitary wave to exist in $(2+1)$ dimensions, as confirmed by a careful examination of the numerical scheme and its solutions. Finally, nematic liquid crystals subjected to two different external electric fields can support the same solitary wave, exhibiting a new type of bistability.

1. Introduction

Solitary waves are nonlinear, self-reinforcing waves which occur in a diverse range of applications, from their first observation and modelling in fluids [1,2], to plasma physics [3,4], nonlinear optics [5,6] and biology [7]. Much of this interest has been driven by certain solitary wave equations, including the ubiquitous Korteweg–de Vries (KdV), the nonlinear Schrödinger (NLS) and the Sine–Gordon equations, all having exact solutions of their initial value problems via the inverse scattering method [8,9]. For such equations, the solitary waves are termed solitons due to their particle-like behaviour of interacting without change of form, the only evidence of the collision being a quantifiable phase shift [9]. Solitary waves have found widespread interest in nonlinear optics due to their ability to propagate without change of shape and robustly in the presence of perturbations such as noise [6]. The present work is concerned with solitary waves in a specific family of nonlinear optical media referred to as nematic liquid crystals [10]. Nematic liquid crystals (NLC) are able to support spatial optical solitary waves, so-called nematicons [11,12]. Nematicons are not exact solitons as they do not interact cleanly, with diffractive radiation (continuous spectrum) shed upon their interaction with one another and their tendency to mutually attract and merge [13].

Physically, nematicons arise from a polarized beam of (laser) light propagating in a nematic liquid crystal (NLC) cell, as nematic liquid crystals are positive uniaxial fluids with a self-focusing reorientational response to light intensity [14,15]. The governing,

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"nematicon", equations of this system are a coupled set of partial differential equations [16]. The electric field of the light beam obeys a nonlinear Schrödinger (NLS)-type equation, with the NLC reorientational response governed by an elliptic Poisson-type equation with an inhomogeneity, or forcing, proportional to the intensity of the optical field [11,12,16]. The electric field equation can also be thought of as Schrödinger's equation from quantum mechanics with a potential which depends on the medium response to light. A nematicon can form as the refractive index disturbance induced by the NLC response increases with the beam power, so that nonlinear self-focusing can balance linear diffraction and yield self-trapping [11,12,16,17]. To date, there are no known exact nematicon solutions of these governing equations.

While the specific context of this work is solitary waves in nematic liquid crystals, the system of equations governing nematicons is widely applicable in various physical contexts. The same equations apply to optical solitary waves in thermal nonlinear media [18], such as lead glasses [19–21] and some photorefractive crystals [22,23]. The Schrödinger–Newton equations from the theory of quantum gravitation [24] have the same form as the nematicon equations when a steady solution is sought and there is no initial orientation bias of the NLC molecules. Three dimensional radially symmetric solutions of the Schrödinger–Newton equations which have solitary wave-like form have been proved to exist [25], although such solutions are not directly applicable to solitary waves in nematic liquid crystals. A system similar to the nematicon equations also arises in α models of fluid turbulence [26,27]. With reference to steady solitary waves, the equations governing nematicons also describe two colour solitary waves in parametric media [17,28,29] and it will be shown in the present work that they are further related to those for reaction–diffusion fronts governed by Fisher's equation [30] and those for a self-gravitating gas in astrophysics [31].

In optics, the nematicon equations are said to form a nonlinear, nonlocal system [11,12,17]. Physically, nonlocality is understood in the sense that the NLC response far exceeds the spatial extent of the optical forcing [11,12,17]. However, it will be shown in the present work that the response of the nematic to the optical beam is more involved than this and that the nonlocal response of NLC is properly described by the ellipticity of the equation governing its response, whose solution depends on the entire domain [32].

Due to the lack of exact nematicon solutions, the system governing nematicons, the nematicon equations, have been studied using numerical, approximate and variational methods [12,33,34]. One such general approach, the Snyder–Mitchell model, assumed that the medium response to the beam is substantially wider than the beam itself, the so-called highly nonlocal limit [35]. In this limit the material response is modelled by a quadratic potential, heuristically justified by how much of the medium is "felt" by the optical beam. This reduces the nematicon equations to that for a quantum harmonic oscillator, implying a Gaussian solitary wave profile [17]. The opposite limit, the local limit in which the optical beam and the NLC response have similar widths, has also been investigated in $(1+1)$ dimensions and solitary wave solutions found as a perturbation series [36]. Approximate solutions in $(1+1)$ and $(2+1)$ dimensions have been found for quadratic solitary waves in χ^2 media [28]. These approximate solutions also apply to steady nematicons since the equations governing steady nematicons and quadratic solitary waves are the same [17,28,29]. In the context of solitary waves in nonlinear optics, variational approximations have been found to give useful results when suitable trial functions for the solitary wave profile are assumed [37]. For cases in which the solitary wave evolves, the key to the choice of suitable trial functions is the addition of suitable terms which allow this evolution. One of the most successful of these is the "chirp" method of Anderson [38] for which the solitary wave has a linear chirp added to its frequency. An alternative variational formulation adds a flat shelf under the evolving solitary wave which matches to the radiation shed as it propagates [39,40]. The latter method has the advantage that the additional shed radiation allows the solitary wave to evolve to a steady state, which is not the case for the chirp method as it has not been possible to incorporate radiative loss using this method.

In the present work, isolated, exact nematicon solutions will be found in both $(1+1)$ and $(2+1)$ dimensions. In $(1+1)$ dimensions the exact solution is similar to the soliton solution of the KdV equation [2]. In $(2+1)$ dimensions it is related to the solution of Abel's equation [41]. These exact solutions are isolated as they have no free parameters. Variational approximations are then found to the nematicon solution using sech, sech^2 and Gaussian trial functions. These approximations are then compared with numerically calculated nematicons obtained using the imaginary time evolution method (ITEM) [42]. It will be shown that the hyperbolic secant trial functions give the best agreement with the numerical solutions for parameter values which lie within the experimental range. The Gaussian trial function shows significant disagreement with the numerical solutions, even though this is the predicted soliton profile for very large nonlocality [35]. The reasons for this will be discussed. Moreover, solutions of the nematicon equations can exhibit the same solitary wave profile in different operating regimes of the NLC sample. The implications of this type of bistability will be discussed. Finally, it will be shown that there is a minimum power at which a nematicon can form, with good agreement between the numerical and the variational minimum values.

2. Governing equations

Consider the propagation of a polarized beam of light inside and along a planar cell filled with nematic liquid crystals (NLC), fluid dielectrics with optical birefringence and long range orientational order [10]. The propagation direction is taken to be z and the y direction is taken as the direction of linear polarization of the electric field of the input light. The x coordinate completes the orthogonal coordinate triad. The NLC optic axis (or molecular director) is initially orthogonal to the electric field of the beam. To eliminate the resulting reorientation threshold, the optical Freédericksz transition [10], a low frequency electric field (voltage) is externally applied in the y direction in order to pre-tilt the NLC molecules at a finite angle θ_0 to the z direction in the (y, z) plane [43]. The electric field of the optical beam can then rotate the light-induced molecular dipoles (nematic molecules) by an extra angle θ, so that the molecular director makes a total angle $\theta_0 + \theta$ to the direction of the beam wavevector (taken collinear with z). In this manner, milliwatt power (squared L^2 norm of the electric field) light beams can generate an optical solitary wave, a nematicon [11,12,43], by increasing the extraordinary refractive index n_e [2] of the uniaxial according to

$$n_e = \left(\frac{\cos^2(\theta_0 + \theta)}{n_\perp^2} + \frac{\sin^2(\theta_0 + \theta)}{n_\parallel^2} \right)^{-1/2}, \tag{1}$$

where $n_\parallel$ and $n_\perp$ are the refractive indices for electric fields parallel and normal to the director, respectively. Assuming $\theta_0 \approx \pi/4$ in order to enhance the reorientational response [44], the non-dimensional equations governing the propagation of the beam through the NLC cell in the slowly varying envelope, paraxial approximation, are [11,12,17,34,45]

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\nabla^2 u + u \sin 2\theta = 0, \tag{2}$$

$$\nu \nabla^2 \theta - q \sin 2\theta = -2|u|^2 \cos 2\theta. \tag{3}$$