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Homoclinic complexity in the localised buckling of an extensible conducting rod in a uniform magnetic field

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HIGHLIGHTS

- Studies localised post-buckling behaviour of a rod in a magnetic field.
- Magnetic buckling is governed by multiple Hamiltonian–Hopf bifurcations.
- Multiplicity and shape of localised solutions are different from non-magnetic rods.
- Find delocalisation–relocalisation behaviour of solutions as parameters are varied.

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ABSTRACT

We study the localised buckling of an extensible conducting rod subjected to end loads and placed in a uniform magnetic field. The trivial straight but twisted rod is described by a fixed point of a four-dimensional Hamiltonian system of equations previously shown to be chaotic. Localised solutions are given by homoclinic orbits to this fixed point and we explore the spatial complexity of localised rod configurations by means of shooting and parameter continuation methods that exploit the reversibility of the system of equations. Unlike in localisation studies of non-magnetic rods we find that for certain parameter values multiple Hamiltonian–Hopf bifurcations occur. Where these collide as parameters are varied, solutions exhibit delocalisation–relocalisation behaviour. Our results predict buckling instabilities and post-buckling behaviour of rods under combined mechanical and magnetic loads, which are relevant for electrodynamic space tethers and potentially for conducting nanowires in future electromechanical devices.

1. Introduction

The buckling of an elastic conducting wire in a magnetic field is a classical problem in magnetoelasticity [1]. Wolfe was the first to give a rigorous bifurcation analysis of this problem in the case of a uniform magnetic field directed parallel to the undeformed wire. He first considered a string model for the wire [2], while in subsequent work he considered the more complicated case of a rod model [3] (a string is here understood to be a perfectly flexible elastic wire, while a rod has resistance to bending and twisting). The work on rods was extended to a post-buckling study in [4], where also whirling solutions were considered. In all these studies of finite-length rods an infinite number of coiled solutions was found to bifurcate from the straight (and untwisted) wire in pitchfork bifurcations at critical values of the magnetic field (or, equivalently, the electric current in the rod).

In a separate line of research, in [5] one of us (GH) showed that for a transversely isotropic inextensible and unshearable rod the equilibrium equations are completely integrable. In a follow-up paper [6] it was shown that if the inextensibility/unshearability assumption is dropped one integral (conserved quantity) is lost and numerical evidence, in the form of Poincaré plots, was given of complicated (chaotic) rod configurations. A reduced three-degrees-of-freedom canonical system of equations was also derived from the original nine-dimensional non-canonical system of equilibrium equations by using remaining (Casimir) invariants of the equations. In [7] this reduced system of equations was then used in a Melnikov analysis to prove that the problem of an isotropic extensible and shearable rod in a uniform magnetic field is indeed chaotic, in the sense of exhibiting horseshoe dynamics. From general dynamical systems theory this suggests the existence

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of a multitude of (multi-pulse) homoclinic orbits. Here we investigate the existence, multiplicity and bifurcation of these homoclinic orbits and what the homoclinic complexity means for the post-buckling behaviour of conducting elastic wires placed in a magnetic field.

The study of a conducting rod in a magnetic field is of interest for stability problems of electrodynamic space tethers, i.e., conducting cables that exploit the earth's magnetic field to generate thrust and drag (Lorentz) forces for manoeuvring [4,8]. Long tethers, tensioned by the earth's gravity gradient, may buckle under the combined action of magnetic and elastic (twisting) forces. For such long tethers the energetically preferred post-buckling states will be localised ones [9], described by homoclinic orbits of the equilibrium equations.

Homoclinic orbits have been studied in other nonintegrable perturbations of the classical integrable Kirchhoff rod. In [10] an anisotropic rod was considered, while in [11] a rod with intrinsic curvature was considered. In both cases localised buckling was found to be described by a Hamiltonian–Hopf bifurcation of a fixed point where a number of homoclinic orbits bifurcate, replacing the single homoclinic orbit of the Kirchhoff rod [12]. The fixed point represents the trivial state of a straight but end-loaded rod. The precise number of bifurcating homoclinic orbits depends on the (reversing) symmetry properties of the system of equations. We find similar behaviour in the present problem with homoclinic orbits bifurcating from a Hamiltonian–Hopf bifurcation. To investigate these homoclinic orbits we can therefore use the same combination of a shooting and a parameter continuation method used in [10], in both cases exploiting the (single) reversing symmetry of the governing equations. However, there are also differences. We find a richer structure of homoclinic orbits than in previous studies, with various families of (multi-pulse) orbits. Unlike in previous studies, for certain values of the problem parameters there are two critical Hamiltonian–Hopf points. These may collide as a secondary bifurcation parameter is varied and this has implications for the stability of the rod against combined mechanical and magnetic loads. We also find that, since post-buckled solutions incur a distributed Lorentz force in addition to an end force and end twisting moment, the localised rod configurations are different from classical solutions in that their ends are generally not co-axial.

The rest of the paper is outlined as follows. Section 2 recalls the equilibrium equations for an extensible and shearable rod in a uniform magnetic field and discusses limiting cases where analytical results can be obtained. Section 3 briefly introduces the shooting method and presents structured families of homoclinic orbits computed using this method. The bifurcation behaviour of these solutions is explored by parameter continuation in Section 4. Concluding remarks are made in Section 5.

2. The mathematical model

2.1. Kinematic equations

In Cosserat theory [13] a rod is characterised by a space curve $\boldsymbol{r}(s)$, describing the centreline of the rod, and an attached right-handed orthonormal triad of directors $\{\boldsymbol{d}_1(s), \boldsymbol{d}_2(s), \boldsymbol{d}_3(s)\}$, describing the varying orientation of the cross-section. Here s is undeformed arclength and $\boldsymbol{d}_1(s)$ and $\boldsymbol{d}_2(s)$ are pointing along the two orthogonal (material) principal axes in the rod's cross-section at s (see Fig. 1).

On introducing a right-handed orthonormal frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ fixed in space we can write

$$\boldsymbol{d}_i = \boldsymbol{R}\boldsymbol{e}_i \tag{1}$$

where $\boldsymbol{R}$ is a rotation matrix. $\boldsymbol{e}_3$ is chosen in the direction of the (uniform) magnetic field. We parametrise $\boldsymbol{R}$ in terms of Euler angles $(\theta,\ \psi,\ \phi)$ as given in Box I.

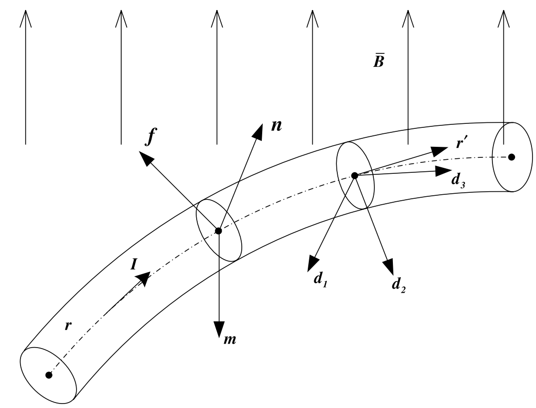

Fig. 1. Schematic diagram of a Cosserat rod.

Differentiating (1) gives

$$\boldsymbol{d}_i' = \boldsymbol{R}'\boldsymbol{e}_i = \boldsymbol{R}'\boldsymbol{R}^{T}\boldsymbol{d}_i =: \boldsymbol{u} \times \boldsymbol{d}_i \tag{2}$$

where $\boldsymbol{u}$ is the first strain vector whose body components $u_i = \boldsymbol{u}\cdot\boldsymbol{d}_i$ are the curvatures ($i = 1, 2$) and the twist ($i = 3$) of the rod, which can be expressed as

$$u_i = \frac{1}{2}\varepsilon_{ijk}\boldsymbol{d}_j' \cdot \boldsymbol{d}_k \tag{3}$$

where ε_{ijk} is the usual antisymmetric symbol. The second strain vector v is given by

$$\boldsymbol{r}' = \boldsymbol{v}. \tag{4}$$

Its body components $v_i = \boldsymbol{v} \cdot \boldsymbol{d}_i$ are strains associated with shear ($i = 1, 2$) and stretch ($i = 3$) [6,13].

2.2. Equilibrium equations

The equilibrium equations for the resultant force $\boldsymbol{n}$ and moment $\boldsymbol{m}$ in the rod (integrated over the cross-section and acting at the centreline) are [13]

$$\boldsymbol{n}' + \boldsymbol{f} = \boldsymbol{0} \tag{5}$$

$$\boldsymbol{m}' + \boldsymbol{r}' \times \boldsymbol{n} + \boldsymbol{l} = \boldsymbol{0} \tag{6}$$

where $\boldsymbol{f}$ and $\boldsymbol{l}$ are external distributed loads. In our case, the only distributed load we shall consider is that due to a magnetic field, in which case $\boldsymbol{f}$ is given by the Lorentz force, while $\boldsymbol{l} = \boldsymbol{0}$. Assuming the rod to be conducting and to carry a current $\boldsymbol{I} = I\boldsymbol{r}'$ the Lorentz force experienced when placed in a magnetic field $\bar{\boldsymbol{B}}$ is

$$\boldsymbol{f} = \boldsymbol{I} \times \bar{\boldsymbol{B}} = I\boldsymbol{r}' \times \bar{\boldsymbol{B}} = \lambda\boldsymbol{r}' \times \boldsymbol{e}_3 \tag{7}$$

where $\lambda = I\left|\bar{\boldsymbol{B}}\right|$ [3].

2.3. Constitutive relations

We assume the rod to be hyperelastic, i.e., we assume that there is a strain energy density function $W = W(u_i, v_i)$ such that the body components $m_i = \boldsymbol{m} \cdot \boldsymbol{d}_i$ of the moment $\boldsymbol{m}$ and $n_i = \boldsymbol{n} \cdot \boldsymbol{d}_i$ of the force $\boldsymbol{n}$ are given by

$$m_i = \frac{\partial W}{\partial u_i} \qquad n_i = \frac{\partial W}{\partial v_i}.$$

We will consider the important special case of linear elasticity, where the strain energy is quadratic in the strains:

$$W(u_i, v_i) = \frac{1}{2}B_1u_1^2 + \frac{1}{2}B_2u_2^2 + \frac{1}{2}Cu_3^2 + \frac{1}{2}G_1v_1^2 + \frac{1}{2}G_2v_2^2 + \frac{1}{2}K(v_3 - 1)^2. \tag{8}$$