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ASYMPTOTIC FORMULAE FOR THE BLOCH EIGENVALUES NEAR PLANES OF DIFFRACTION

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In this paper we obtain asymptotic formulae of arbitrary order for the Bloch eigenvalue of the periodic Schrödinger operator $-\Delta + q(x)$, of arbitrary dimension, when the corresponding quasi-momentum lies near planes of diffraction.

Keywords: Bloch eigenvalue, Schrödinger operator, perturbation.

1. Introduction

In this paper we consider the operator

$$L(q) = -\Delta + q(x), \qquad x \in \mathbb{R}^d, \qquad d \geq 2 \tag{1}$$

with a periodic (relative to a lattice Ω) potential $q(x) \in W_2^s(F)$, where

$$s \geq \frac{1}{2}d^2 + 6d + 4, \qquad F \equiv \mathbb{R}^d / \Omega$$

is a fundamental domain of Ω. Without loss of generality it can be assumed that the measure $\mu(F)$ of F is 1 and $\int_F q(x)dx = 0$. Let $L_t(q(x))$ be the operator generated in F by (1) and the conditions

$$u(x + \omega) = e^{i(t,\omega)}u(x), \qquad \forall \omega \in \Omega,$$

where $t \in F^\star \equiv \mathbb{R}^d/\Gamma$ and Γ is the lattice dual to Ω, that is, Γ is the set of all vectors $\gamma \in \mathbb{R}^d$ satisfying $(\gamma, \omega) \in 2\pi\mathbb{Z}$ for all $\omega \in \Omega$. It is well known that (see [1]) the spectrum of the operator $L_t(q)$ consists of the eigenvalues $\Lambda_n(t)$ $(n = 1, 2, \dots)$ corresponding to the Bloch functions $\Psi_{n,t}(x)$:

$$L_t(q)\Psi_{n,t}(x) = \Lambda_n(t)\Psi_{n,t}(x). \tag{2}$$

In the case $q(x) = 0$ these eigenvalues and eigenfunctions are $|\gamma + t|^2$ and $e^{i(\gamma+t,x)}$ for $\gamma \in \Gamma$:

$$L_t(0)e^{i(\gamma+t,x)} = |\gamma + t|^2 e^{i(\gamma+t,x)}. \tag{3}$$

In [5–9] for the first time the eigenvalues $|\gamma + t|^2$, for big $\gamma \in \Gamma$, were divided into two groups: non-resonance ones (roughly speaking, if $\gamma + t$ is far from the diffraction planes) and resonance ones (if $\gamma + t$ is near a diffraction plane) and for the perturbations of each group various asymptotic formulae were obtained. To give the precise definition of the non-resonance and resonance eigenvalue $|\gamma + t|^2$ of order ρ^2 (written as $|\gamma + t|^2 \sim \rho^2$, for definiteness suppose $\gamma + t \in R(\frac{3}{2}\rho)\backslash R(\frac{1}{2}\rho)$), where $R(\rho) = \{x \in \mathbb{R}^d : |x| < \rho\}$) for a big parameter ρ we write the potential $q(x) \in W_2^s(F)$ in the form

$$q(x) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} e^{i(\gamma_1, x)} + O(\rho^{-p\alpha}), \tag{4}$$

where

$$p = s - d, \qquad \alpha = \frac{1}{d + 11}, \qquad q_\gamma = (q(x), e^{i(\gamma, x)}) = \int_F q(x) e^{-i(\gamma, x)} dx.$$

$\Gamma(\rho^\alpha) = \{\gamma \in \Gamma : 0 < |\gamma| < \rho^\alpha)\}$, and the relation $|\gamma + t|^2 \sim \rho^2$ means that there exist constants c_1 and c_2 such that $c_1\rho < |\gamma + t| < c_2\rho$ (here and in subsequent relations we denote by c_i $(i = 1, 2, \dots)$ the positive, independent of ρ constants whose exact values are inessential). Note that $q(x) \in W_2^s(F)$ means that $\sum_\gamma |q_\gamma|^2(1 + |\gamma|^{2s}) < \infty$. If $s \geq d$, then

$$\sum_\gamma |q_\gamma| < c_3, \qquad \sup \left| \sum_{\gamma \notin \Gamma(\rho^\alpha)} q_\gamma e^{i(\gamma, x)} \right| \leq \sum_{|\gamma| \geq \rho^\alpha} |q_\gamma| = O(\rho^{-p\alpha}), \tag{5}$$

i.e. (4) holds. It follows from (5) that the influence of $\sum_{\gamma \notin \Gamma(\rho^\alpha)} q_\gamma e^{i(\gamma, x)}$ on the eigenvalue $|\gamma + t|^2$ is $O(\rho^{-p\alpha})$. In [7–9] in order to observe the influence of the trigonometric polynomial $P(x) = \sum_{\gamma \in \Gamma(\rho^\alpha)} q_\gamma e^{i(\gamma, x)}$ on the eigenvalue $|\gamma + t|^2$ we used the formula

$$(\Lambda_N - |\gamma + t|^2) b(N, \gamma) = (\Psi_{N,t}(x) q(x), e^{i(\gamma + t, x)}), \tag{6}$$

where $b(N, \gamma) = (\Psi_{N,t}(x), e^{i(\gamma + t, x)})$, which is obtained from Eq. (2) by multiplying by $e^{i(\gamma + t, x)}$ and using (3). We say that (6) is the binding formulae for $L_t(q)$ and $L_t(0)$, since it connects the eigenvalues and eigenfunctions of $L_t(q)$ and $L_t(0)$. Introducing the expansion (4) of $q(x)$ into (6) we get

$$(\Lambda_N - |\gamma + t|^2) b(N, \gamma) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} b(N, \gamma - \gamma_1) + O(\rho^{-p\alpha}). \tag{7}$$

If Λ_N is close to $|\gamma + t|^2$ and $\gamma + t$ does not belong to any of the sets

$$V_{\gamma_1}(\rho^{\alpha_1}) \equiv \{x \in \mathbb{R}^d : ||x|^2 - |x + \gamma_1|^2| < \rho^{\alpha_1}\} \cap (R(3\rho/2)\backslash R(\rho/2)) \tag{8}$$

for $\gamma_1 \in \Gamma(\rho^\alpha)$, where $\alpha_1 = 3\alpha$, that is, $\gamma + t$ are far from the diffraction planes $\{x \in \mathbb{R}^d : |x|^2 - |x + \gamma_1|^2 = 0\}$ for $\gamma_1 \in \Gamma(\rho^\alpha)$, then

$$||\gamma + t|^2 - |\gamma - \gamma_1 + t|^2| \geq \rho^{\alpha_1}, \qquad |\Lambda_N - |\gamma - \gamma_1 + t|^2| > \frac{1}{2}\rho^{\alpha_1} \tag{9}$$