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On the coupling between an ideal fluid and immersed particles

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HIGHLIGHTS

- Interpolation methods for velocity fields are principal connections.
- The horizontal Lagrange–Poincaré equations yield particle methods for fluids.
- Higher-order interpolation methods are derived by higher-order isotropy subgroups.
- Higher-order isotropy groups yield finite-dimensional circulation theorems.

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ABSTRACT

In this paper, we present finite-dimensional particle-based models for fluids which respect a number of geometric properties of the Euler equations of motion. Specifically, we use Lagrange–Poincaré reduction to understand the coupling between a fluid and a set of Lagrangian particles that are supposed to simulate it. We substitute the use of principal connections in Cendra et al. (2001) [13] with vector field valued interpolations from particle velocity data. The consequence of writing evolution equations in terms of interpolation is two-fold. First, it provides estimates on the error incurred when interpolation is used to derive the evolution of the system. Second, this form of the equations of motion can inspire a family of particle and hybrid particle–spectral methods, where the error analysis is "built in". We also discuss the influence of other parameters attached to the particles, such as shape, orientation, or higher-order deformations, and how they can help us achieve a particle-centric version of Kelvin's circulation theorem.

1. Introduction

Particles are capable of carrying a variety of information such as position, shape, and orientation. Moreover, when this data is described with finitely many numbers, we may consider including it as input for a computer simulation. Given this point of view on particles, we seek to understand, from a geometric perspective, how a set of Lagrangian particles can be used as a computational device to numerically simulate an ideal fluid. We will explore this idea by applying Lagrange–Poincaré reduction to the exact equations of motion. The horizontal Lagrange–Poincaré equation can be used to inspire a family of particle methods. Specifically, given an ideal, homogeneous, inviscid, incompressible fluid on a Riemannian manifold M with smooth boundary ∂M, oriented by the associated Riemannian volume, the configuration space may be described by the group of volume-preserving diffeomorphisms, $\mathrm{SDiff}(M)$, and the exact equations of motion for an ideal fluid are the L^2-geodesic equations as described in [1]. Throughout the paper we shall assume that there is a Hodge decomposition; for compact boundaryless manifolds, this is standard (see, e.g., [2]); for compact manifolds with a boundary, this holds in the case of ∂-manifolds (i.e., the manifold is, in addition, complete as a metric space; see [3]); for non-compact manifolds, this holds in function spaces with enough decay at infinity (for $\mathbb{R}^n$, see, e.g., [4–6]; and if the manifold has a boundary, see [3]).

If $\odot$ is an N-tuple of distinct points in M, let

$$G_\odot := \{\psi \in \mathrm{SDiff}(M) \mid \psi(\odot) = \odot\}$$

be the isotropy subgroup of the natural action of $\mathrm{SDiff}(M)$ on M. The particle relabeling symmetry of an ideal fluid allows us to project the equations of motion onto the quotient space $T\mathrm{SDiff}(M)/G_\odot$. Upon choosing an interpolation method, i.e., a

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means of interpolating a smooth vector field between the particles (see Fig. 1), we obtain an isomorphism to a direct sum of vector bundles, $TX \oplus \tilde{\mathfrak{g}}_{\odot}$. Here, the base manifold, X, is the configuration space of point particles in M, and TX is the tangent bundle of X (the state or velocity phase space). The second component, $\tilde{\mathfrak{g}}_{\odot}$, is a vector bundle over X whose fiber over $x \in X$ is the infinite-dimensional vector space of vector fields which vanish at the particle locations described by x. The vector bundle $\tilde{\mathfrak{g}}$ stores the residual of an estimated spatial velocity field u obtained by an interpolation velocity data at a finite set of points. More generally, we can consider reducing by the subgroup

$$G_{\odot}^{(k)} := \{\psi \in \mathrm{SDiff}(M) \mid T_{\odot}^{(k)}\psi = T_{\odot}^{(k)} id\}.$$

The resulting Lagrange–Poincaré equations occur on a direct sum $TX^{(k)} \oplus \tilde{\mathfrak{g}}_{\odot}^{(k)}$, where $X^{(k)}$ is the configuration manifold of a more sophisticated type of particle which carries extra data such as orientation and shape; see Section 4 for the definitions of all these objects. The equations on $TX^{(k)} \oplus \tilde{\mathfrak{g}}_{\odot}^{(k)}$ describe the coupling between a fluid and this sophisticated type of particle in terms of interpolation methods. The dynamics on the $TX^{(k)}$ component suggests a new class of higher-order spatially accurate particle methods.

1.1. Organization and main contributions

To understand the intent of this paper, it helps to explain what we mean by an *interpolation method*. While a formal definition will be given in Section 2.3, the idea is fairly simple. Given N particles in M equipped with various data (e.g., position, velocity, orientation, higher-order deformations), an interpolation method is a rule which produces a vector field on M that is consistent with this data (see Fig. 1). Using the concept of an interpolation method, this paper accomplishes the following.

(1) For each interpolation method, we construct an isomorphism between the quotient space $(T\mathrm{SDiff}(M))/G_{\odot}$ and the vector bundle $TX \oplus \tilde{\mathfrak{g}}_{\odot}$ (Proposition 2.3).
(2) We derive equations of motion on $TX \oplus \tilde{\mathfrak{g}}_{\odot}$ for an arbitrary $G_{\odot}$-invariant Lagrangian on $\mathrm{SDiff}(M)$ (i.e., the Lagrange–Poincaré equations, Theorem 3.1).
(3) We generalize these constructions to higher-order interpolation methods. The resulting equations describe a family of particle methods wherein the particles carry extra data such as orientation, shape, and higher-order deformations (Theorem 5.1 and Corollary 5.1).
(4) The numerical methods of Corollary 5.1 exhibit a particle-centric analog of Kelvin's circulation theorem (Theorem 5.5).
(5) We illustrate how first-order interpolation methods induce particle methods which are related to the vortex blob method.

In particular, these goals are accomplished as follows. In Section 2, we establish our notation and review the notion of a generalized connection (also called an *Ehresmann connection*; see [7]) as described in [8]. In Section 3, we carry out the reduction process by the isotropy subgroup of a finite set of particles for an ideal incompressible homogeneous inviscid fluid. The necessary Lin contraints are addressed in the Appendix. In Section 4, we discuss reduction by higher-order isotropies in order to discuss particles with orientation, shape, and other attributes. In certain circumstances, this additional information produces particle methods which exhibit conservation laws found in the exact dynamics on $\mathrm{SDiff}(M)$. In Section 5, we formulate a family of particle methods induced by an interpolation method and discuss some implications for the error analysis of these methods. We conclude that it is possible to construct hybrid particle–spectral methods for fluids within this family. Moreover, we show that the vortex blob algorithm fits within this family of methods and that the horizontal

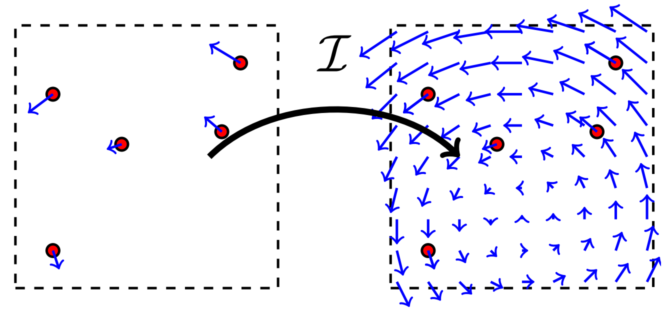

Fig. 1. Schematic representation of an interpolation method.

equations are a guide for corrections that allow for the deformation of vortex blobs. We close with Section 6, where we summarize how to extend these constructions to complex fluids, turbulence models, and the template matching problems which occur in medical imaging.

1.2. Previous work

It was shown in [1] that the Euler equations of motion for an ideal, homogeneous, inviscid, incompressible fluid on an oriented Riemannian manifold M with smooth boundary are the spatial (or Eulerian) representation of the geodesic equations on the group of volume-preserving diffeomorphisms, $\mathrm{SDiff}(M)$. This observation gave rise to a new perspective on fluid mechanics which lead to many developments, notably the proof of well posedness [9] and various extensions ranging all the way to charged fluids, magnetohydrodynamics, and even complex fluids with advected parameters (see, e.g., [10,11]). All of these systems are Lagrangian on the tangent bundle of groups of diffeomorphisms of a Riemannian manifold M. Additionally, these theories utilize the particle relabeling symmetry of the system to perform Euler–Poincaré reduction and thus bring the dynamics to the Lie algebra of this group [12, Chapter 13].

As a result of this $\mathrm{SDiff}(M)$ symmetry, we may consider reducing by subgroups of $\mathrm{SDiff}(M)$. This would be a special case of Lagrange–Poincaré reduction introduced and developed in [13]. In particular, we may consider reducing by isotropy groups of a set of points in M. Such an approach is already mentioned in [14] for vortex dynamics and in [15] for the purpose of landmark matching problems; see also the references cited therein. However, to the best of our knowledge, Lagrange–Poincaré reduction has not been performed on such systems in the framework of [13].

2. Preliminary material

Before introducing our contributions, we review generalized connections and volume-preserving diffeomorphisms and prove a few important theorems.

2.1. Generalized connections

In this section, we introduce the notion of a generalized connection, as presented in [8], and prove some useful propositions for the purpose of this paper.

Definition 2.1. Let $\pi_E : E \to M$ be a vector bundle, and let $\tau_E : TE \to E$ be the tangent bundle of E. The *vertical bundle* is the vector bundle $\pi_{V(E)} : V(E) \to E$, where $V(E) := \mathrm{kernel}(T\pi_E)$ and $\pi_{V(E)} := \tau_E|_{V(E)}$.