APPLICATION OF NONLINEAR BRATU'S EQUATION IN TWO AND THREE DIMENSIONS TO ELECTROSTATICS

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In this work, we focus our attention on some solutions to an electrostatic and plasma problem that consists in solving Poisson's equation. The latter is related to an old problem known as Bratu's problem or Bratu's equation. We present some solutions to this equation and apply them to problems encountered in electrostatics and plasma physics.

Keywords: electrostatics, plasma, potential, Poisson equation, nonlinear equation, Liouville equation, Bratu problem.

1. Introduction

Nonlinear problems are of interest to engineers, physicists, mathematicians and many other scientists because most phenomena in nature are inherently nonlinear. In mathematics, a nonlinear phenomenon is then described by a nonlinear equation or by a system of nonlinear equations. Among these equations, Poisson's equation or Bratu's and Liouville's equations received much attention in recent years. These equations are extensively used in astrophysics [1] and combustion theory [2]. They are also devoted to describe: the Gaussian curvature problem in Riemannian geometry [3], the mean field limit of vortices in Euler flows [4], the Onsager formulation in statistical mechanics [5], the Keller–Siegel system of chemotaxis [6] and the Chern–Simon–Higgs gauge theory [7, 8]. These equations have many other physical applications such as in chemical reactor theory, in radiative heat transfer, in electrostatics, in fluid mechanics and in the theory of the universe expansion. Because of its simplicity, Bratu's equation is widely used to test nonlinear eigenvalue solvers [9]. On the other hand, Liouville's equation is mathematically appealing since it has an interesting solution structure.

In Section 2, we shall construct an integral equation governing the effective potential which is a nonlinear integral equation. This potential describes the energy interaction between an electric charge and the rest of the electrostatic system consisting of negative and positive charges (electrons and ions commonly defined as plasma). Hence we derive the differential equation governing this effective potential. It happens that this last equation is nonlinear and furthermore is the Bratu equation. In Section 3, we give some exact solutions of our problem in two dimensions. Section 4 is devoted to give some useful applications in plasma physics in three dimensions. We close this paper by a conclusion.

2. Integral and differential equations for the effective energy potential

Let us consider a medium consisting of electrons and a continuous background of neutralizing positive electrical charges. At first, the distribution of electrons is that of Maxwell–Boltzmann governing the equilibrium state of the electrons system. If we place a positive ion of charge Ze (called the test charge or impurity) at the coordinates origin, the system is disturbed and after a certain time t it will reach a new equilibrium state described by a novel distribution of electrons over the space around the charge Ze. The latter is determined through the potential energy of an electron located at a distance r from the test charge Ze when the system has reached this novel equilibrium state. This potential energy is built as a sum of three contributions,

$$V(r) = V_{\mathrm{ie}}(r) + V_{\mathrm{ee}}(r) + V_{\mathrm{ef}}(r)\,, \tag{1}$$

where $V_{\mathrm{ie}}(r)$ is the potential energy of ion–electron interaction (the ion is the test charge), $V_{\mathrm{ee}}(r)$ is the interaction energy of the electron with all other electrons and $V_{\mathrm{ef}}(r)$ is the interaction energy of the electron with the continuous neutralizing background of ions. The electron–electron interaction is that of the Coulomb potential energy e^2/r such as the potential energy $V_{\mathrm{ee}}(r)$, in the mean field approximation is equal to

$$V_{\mathrm{ee}}(r) = e^2 \int f(\overrightarrow{r'}, \overrightarrow{p})\ \frac{1}{|\overrightarrow{r} - \overrightarrow{r'}|}\ d\overrightarrow{p}^{\,3} d\overrightarrow{r'}^{\,3}, \tag{2}$$

where

$$f(\overrightarrow{r}, \overrightarrow{p}) = \frac{N}{\Xi}\left(\frac{m\beta}{2\pi}\right)^{3/2} \exp\left[-\beta\left(\frac{\overrightarrow{p}^{\,2}}{2m} + V(r)\right)\right] \tag{3}$$

is the Maxwell–Boltzmann distribution and N is the total number of electrons and Ξ is the volume of the system, whereas the potential energy of the electron with the positive background neutralizing charge is given by

$$V_{\mathrm{ef}}(r) = -n_{\mathrm{e}} e^2 \int \frac{1}{|\overrightarrow{r} - \overrightarrow{r'}|} d\overrightarrow{r'}^{\,3}. \tag{4}$$