HAMILTON–JACOBI THEORY IN CAUCHY DATA SPACE

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(*Received April 22, 2015 – Revised July 6, 2015*)

Recently, M. de León *et al.* [8] have developed a geometric Hamilton–Jacobi theory for classical fields in the setting of multisymplectic geometry. Our purpose in the current paper is to establish the corresponding Hamilton–Jacobi theory in the Cauchy data space, and relate both approaches.

Keywords: multisymplectic field theory, Hamilton–Jacobi theory, Cauchy surface, Cauchy data space, covariant formalism.

2010 Mathematics Subject Classification: Primary 70S05, Secondary 70H03, 70H05, 53D12.

1. Introduction

Multisymplectic geometry is the natural arena to develop classical field theories of first order. Indeed, a multisymplectic manifold is a natural extension of symplectic manifolds, and in addition, the canonical models for multisymplectic structures are just the bundles of forms on a manifold in the same vein that cotangent bundles (1-forms) provide the canonical models for symplectic manifolds.

One can exploit this parallelism between classical mechanics and classical field theories but one should go very carefully. Indeed, instead of a configuration manifold, we have now a configuration bundle $\pi E \to M$ such that its sections are the fields (the manifold M represents the space-time manifold). The Lagrangian density depends on the space-time coordinates, the fields and its derivatives, so it is very natural to take the manifold of 1-jets of sections of π, $J^1\pi$, as the generalization of the tangent bundle in classical mechanics; then a Lagrangian density is a fibered mapping $\mathcal{L}J^1\pi \to \Lambda^{m+1}M$ (we are assuming that $\dim M = m+1$). From the Lagrangian density one can construct the Poincaré–Cartan form which gives the evolution of the system.

On the other hand, the spaces of 1- and 2-horizontal $(m+1)$-forms on E with respect to the projection π, denoted respectively by $\Lambda_1^{m+1}E$ and $\Lambda_2^{m+1}E$, are the arena where the Hamiltonian picture of the theory is developed. To be more precise, the phase space is just the quotient

$$\mathcal{M}^o\pi = \Lambda_2^{m+1}E/\Lambda_1^{m+1}E$$

and the Hamiltonian density is a section of $\Lambda_2^{m+1}E \to \mathcal{M}^o\pi$ (the Hamiltonian function H appears when a volume form η on M is chosen, such that $\mathcal{H} = H\,\eta$. The Hamiltonian section $\mathcal{H}$ permits just to pull-back the canonical multisymplectic form of $\Lambda_2^{m+1}E$ to a multisymplectic form on $\mathcal{M}^o\pi$.

Of course, both descriptions are related by the Legendre transform which sends solutions of the Euler–Lagrange equations into solutions of the Hamilton equations. One important difference with the case of mechanics is that now we are dealing with partial differential equations and we lost in principle the integrability. In any case, the solutions in both sides are interpreted as integral sections of Ehresmann connections. For a detailed account of the multisymplectic formalism see [3, 6–8, 10, 11, 14, 15, 18, 19, 22, 24].

The Hamilton–Jacobi problem for a Hamiltonian classical field theory given by a Hamiltonian H consists in finding a family of functions $S^i = S^i(x^i, u^\alpha)$ such that

$$\frac{\partial S^i}{\partial x^i} + H\left(x^i, u^\alpha, \frac{\partial S^i}{\partial u^\alpha}\right) = f(x^i) \tag{1}$$

for some function $f(x^i)$. Here (x^i, u^α) represent bundle coordinates in E and (x^i) are coordinates in M.

In [8] the authors have developed a geometric Hamilton–Jacobi theory in the context of multisymplectic manifolds. The procedure used there is an extension of that used in [1], but now considering Ehresmann connections instead of vector fields as solutions of the Hamilton equations. Let us notice that an alternative approach to the Hamilton–Jacobi theory for classical field theories has been developed in the context of k-symplectic and k-cosymplectic structures (see [9, 12, 13]).

One attempt to develop a combined formalism (k-symplectic and k-cosymplectic) can be found in [16, 17].

As it is well known (see [3, 18]) there is an alternative way to study classical field theories, in an infinite-dimensional setting. The idea is to split the space-time manifold M in the space and time pieces. To do this, we need to take a Cauchy surface, that is, an m-dimensional submanifold N of M such that (at least locally) we have $M = \mathbb{R} \times N$. So, the space of embeddings from N to $\mathcal{M}^o\pi$ is known as the Cauchy space of data for a particular choice of a Cauchy surface. This allows us to integrate the multisymplectic form on $\mathcal{M}^o\pi$ to the Cauchy data space and obtain a presymplectic infinite-dimensional system, whose dynamics is related to the de Donder–Hamilton equations for H.

The aim of the paper is to show how we can "integrate" a solution of the Hamilton–Jacobi problem for H in order to get a solution for the Hamilton–Jacobi problem for the infinite-dimensional presymplectic system.