ELEMENTARY RESULTS FOR THE FUNDAMENTAL REPRESENTATION OF SU(3)

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A general group element for the fundamental representation of SU(3) can always be expressed as a second order polynomial in a Hermitian generating matrix H, with coefficients consisting of elementary trigonometric functions dependent on the sole invariant $\det(H)$, in addition to the group parameter.

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In memoriam Yoichiro Nambu (1921–2015)

Consider an arbitrary 3×3 traceless Hermitian matrix H. The Cayley–Hamilton theorem [1] gives

$$H^3 = I \det(H) + \frac{1}{2} \; H \operatorname{tr}(H^2), \tag{1}$$

and therefore $\det(H) = \operatorname{tr}(H^3)/3$. Note that an H^2 term is absent in the polynomial expansion of H^3 because of the trace condition, $\operatorname{tr}(H) = 0$. Also note, since $\operatorname{tr}(H^2) > 0$ for any nonzero Hermitian H, this bilinear trace factor may be absorbed into the normalization of H, thereby setting the scale of the group parameter space.

We may now write the exponential of H as a matrix polynomial [2, 3]. As a consequence of (1) any such exponential can be expressed as a matrix polynomial second-order in H, with polynomial coefficients that depend on the displacement from the group origin as a "rotation angle" θ.

Moreover, the polynomial coefficients will also depend on invariants of the matrix H. These invariants can be expressed in terms of the eigenvalues of H, of course [2–7]. Nevertheless, while the eigenvalues of H will be manifest in the final result given below, a deliberate diagonalization of H is not necessary. This is true for SU(3), for a normalized H, since there is effectively only one invariant: $\det(H)$. This invariant may be encoded cyclometrically as another angle. Define

$$\phi = \frac{1}{3}\left(\arccos\left(\frac{3}{2}\sqrt{3}\det(H)\right) - \frac{\pi}{2}\right), \tag{2}$$

whose geometrical interpretation will soon be clear. Inversely,

$$\det(H) = -\frac{2}{3\sqrt{3}} \sin(3\phi) . \tag{3}$$

The result for any SU(3) group element generated by a traceless 3×3 Hermitian matrix H is then

$$\exp(i\theta H) = \sum_{k=0,1,2} \left[H^2 + \frac{2}{\sqrt{3}} H \sin(\phi + 2\pi k/3) - \frac{1}{3} I \left(1 + 2\cos(2(\phi + 2\pi k/3))\right) \right] \frac{\exp\left(\frac{2}{\sqrt{3}} i\theta \sin(\phi + 2\pi k/3)\right)}{1 - 2\cos(2(\phi + 2\pi k/3))} \tag{4}$$

where we have set the scale for the θ parameter space by choosing the normalization

$$\mathrm{tr}(H^2) = 2. \tag{5}$$

With this choice, the Cayley–Hamilton result (1) is just [9]

$$H^3 = H + I \det(H). \tag{6}$$

The normalization (5) and the identity (6) are consistent with the Gell-Mann λ-matrices [8].

So expressed as a matrix polynomial, the group element depends on the sole invariant $\det(H)$ in addition to the group rotation angle θ. Both dependencies are in terms of elementary trigonometric functions when $\det(H)$ is expressed as the angle ϕ, whose geometrical interpretation follows immediately from the three eigenvalues of H exhibited in the exponentials of (4). Those eigenvalues are the projections onto three mutually perpendicular axes of a single point on a circle formed by the intersection of the $0 = \mathrm{tr}(H)$ eigenvalue plane with the $2 = \mathrm{tr}(H^2)$ eigenvalue 2-sphere. The angle ϕ parameterizes that circle. Equivalently, the eigenvalues are the projections onto a single axis of three points equally spaced on a circle [10].

Two cases deserve special mention. On the one hand, the Rodrigues formula for SO(3) rotations about an axis $\widehat{n}$, as generated by $j = 1$ spin matrices, is obtained for $\phi = 0 = \det(H)$. Thus

$$\exp(i\theta H)|_{\phi=0} = I + iH \sin\theta + H^2 (\cos\theta - 1) . \tag{7}$$

This is the Euler–Rodrigues result, upon identifying $H = \widehat{n} \cdot \overrightarrow{J}$ (see [11, 12]). It provides an explicit embedding $SO(3) \subset SU(3)$. In fact, (7) is true if H is any *one* of the first seven Gell-Mann λ-matrices [8], or if H is a normalized linear combination of λ_{1-3}, or of λ_{4-7}. However, for generic linear combinations of λ_{1-7}, $\det(H)$ will *not* necessarily vanish, and the general result (4) must be used.

On the other hand,

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{8}$$