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TOP ELEMENT PROBLEM AND MACNEILLE COMPLETIONS OF GENERALIZED EFFECT ALGEBRAS

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Effect algebras (EAs), introduced by D. J. Foulis and M. K. Bennett, as common generalizations of Boolean algebras, orthomodular lattices and MV-algebras, are nondistributive algebraic structures including unsharp elements. Their unbounded versions, called generalized effect algebras, are posets which may have or may have not an EA-MacNeille completion, or cannot be embedded into any complete effect algebra. We give a necessary and sufficient condition for a generalized effect algebra to have an EA-MacNeille completion. Some examples are provided.

Keywords: effect algebra, generalized effect algebra, orthoalgebra, MacNeille completion of a poset, one-element EA-extension of a generalized effect algebra, EA-MacNeille completion of a generalized effect algebra.

1. Introduction and basic definitions

Effect algebras (EAs), introduced by Foulis and Bennett [3], are very suitable generalizations of Boolean algebras in order to describe sets whose elements may be mutually noncompatible or unsharp. Consequently, an effect algebra is a nondistributive generalization including also unsharp elements x for which x and x' (non x) are not disjoint.

The prototype for axiomatic system of partially defined operation $\oplus$, representing parallel measurements of two effects, was the set $\mathcal{E}(\mathcal{H})$ of all self-adjoint linear operators between the null and the identity operators in a complex Hilbert space $\mathcal{H}$. Note that simultaneously an equivalent (in some sense) structure called a D-poset

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has been introduced by Kôpka [9] and Kôpka and Chovanec [10], for studing fuzzy events. Recently it was shown in [13, 18] that the set of all positive linear operators densely defined on an infinite-dimensional complex Hilbert space $\mathcal{H}$ equipped with a partial sum $\oplus$, forms a generalized effect algebra (see [13, 18, 19]).

Generalized effect algebras are unbounded versions of effect algebras introduced by several authors: Foulis and Bennett [3] (cones), Kalmbach and Riečanová [8] (Abelian RI-semigroups and RI-posets), Hedlíková and Pulmannová [5] (cancelative positive partial Abelian semigroups). All these generalizations are mutually equivalent.

In a general algebraic form an effect algebra (generalized effect algebra) is defined as follow.

DEFINITION 1.1 [3]. A partial algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ is called an *effect algebra* if $\mathbf{0}, \mathbf{1}$ are two distinguished elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x, y, z \in E$:

(E1) $x \oplus y = y \oplus x$ if one side is defined,

(E2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(E3) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = \mathbf{1}$ (we put $x' = y$),

(E4) if $x \oplus \mathbf{1}$ is defined then $x = \mathbf{0}$.

We often denote the effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ briefly by E. On every effect algebra E the partial order $\leq$ and partial binary operation $\ominus$ can be introduced as follows:

$$x \leq y \quad \text{and} \quad y \ominus x = z \quad \text{iff} \quad x \oplus z \quad \text{is defined and} \quad x \oplus z = y.$$

If E with the defined partial order is a lattice (a complete lattice) then $(E, \oplus, \mathbf{0}, \mathbf{1})$ is called a *lattice effect algebra* (a *complete effect algebra*).

DEFINITION 1.2. (1) A *generalized effect algebra* $(E, \oplus, \mathbf{0})$ is a set E with an element $\mathbf{0} \in E$ and a partial binary operation $\oplus$ satisfying for any $x, y, z \in E$ the conditions

(GE1) $x \oplus y = y \oplus x$ if one side is defined,

(GE2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(GE3) if $x \oplus y = x \oplus z$ then $y = z$ (cancellation law),

(GE4) if $x \oplus y = \mathbf{0}$ then $x = y = \mathbf{0}$

(GE5) $x \oplus \mathbf{0} = x$ for all $x \in E$.

(2) A binary relation $\leq$ (being a partial order) on E can be defined by

$$x \leq y \quad \text{iff there exists } z \in E, \quad x \oplus z = y.$$

(3) $Q \subset E$ is called a *sub-generalized effect algebra* (*sub-effect algebra*) of E if $\mathbf{0} \in Q$ ($\mathbf{1} \in Q$) and if out of elements $x, y, z \in E$ such that $x \oplus y = z$ at least two are in Q, then all three are in Q.

(4) If elements of a (generalized) effect algebra E are positive linear operators densely defined in an infinite-dimensional complex Hilbert space $\mathcal{H}$ then we call E an *operator (generalized) effect algebra*.