INVERSE VARIATIONAL PROBLEM FOR NONSTANDARD LAGRANGIANS

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In the mathematical physics literature the nonstandard Lagrangians (NSLs) were introduced in an ad hoc fashion rather than being derived from the solution of the inverse problem of variational calculus. We begin with the first integral of the equation of motion and solve the associated inverse problem to obtain some of the existing results for NSLs. In addition, we provide a number of alternative Lagrangian representations. The case studies envisaged by us include (i) the usual modified Emden-type equation, (ii) Emden-type equation with dissipative term quadratic in velocity, (iii) Lotka–Volterra model and (vi) a number of the generic equations for dissipative-like dynamical systems. Our method works for nonstandard Lagrangians corresponding to the usual action integral of mechanical systems but requires modification for those associated with the modified actions like $S = \int_a^b e^{L(x,\dot{x},t)dt}$ and $S = \int_a^b L^{1-\gamma}(x, \dot{x}, t)dt$ because in the latter case one cannot construct expressions for the Jacobi integrals.

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1. Introduction

In the calculus of variations one deals with two types of problems, namely, the direct and inverse problems of mechanics. The direct problem is essentially the conventional one in which one assigns a Lagrangian to the physical system and then computes the equations of motion using the Euler–Lagrange equations. On the contrary, the inverse problem begins with the equation of motion and then derives an expression for the Lagrangian of the system by a strict mathematical procedure [1].

The Lagrangian L of an autonomous differential equation is expressed as $L = T - V$ where T is the kinetic energy of the system modeled by the equation and V the corresponding potential function. In recent years, a new type of Lagrangian functions have been proposed for dissipative-like autonomous differential equations [2]. These do involve neither T nor V, but yet yield the equations of motion via Euler–Lagrange equations. Consequently, these Lagrangians were qualified as nonstandard. It appears that the nonstandard Lagrangians do not have a natural space in the theory of inverse variational problem. In other words, the basic question whether the proposed expressions for NSLs can be obtained from the differential equations which they represent has remained largely unanswered. Our objective in this work is to construct nonstandard Lagrangian functions for a number of physically important one-dimensional (1D) dissipative-like nonlinear dynamical systems by solving the corresponding inverse problems of the calculus of variation.

The inverse problem in classical mechanics was solved by Helmholtz towards the end of the nineteenth century [3]. The use of Helmholtz machinery to solve an inverse problem requires the knowledge of algebro-geometric theories [4]. In the N-dimensional case the Helmholtz conditions are quite difficult to handle but when $N = 1$, they reduce to a single partial differential equation. For 1D problem the Lagrangian always exists and in this work we restrict ourselves to the one-dimensional systems. Recently, Nucci and Leach [5] made use of Jacobi's Last Multipliers [6] to construct Lagrangians for a wide class of second-order differential equations. In this context we identify a solution of the inverse variational problem due to Lopez [7] who provided a method to construct the Lagrangian function for an N-dimensional second-order autonomous differential equation from its first integral. The Lagrangian for the 1D system is given by

$$L(x, v) = v \int^{v} \frac{C(x, \xi)}{\xi^2} d\xi \tag{1}$$

with the velocity v given by

$$v = \frac{dx}{dt}. \tag{2}$$

In (1) C stands for the first integral of the equation that models the system. The expression of the Lagrangian in (1) can be obtained by deriving a first-order partial differential equation for the constant of motion and ultimately relating it to the corresponding Jacobi integral [8]. We shall make use of (1) to construct nonstandard Lagrangian representations for (i) modified Emden-type equations [9] and (ii) Lotka–Volterra equations [10]. In addition, we shall adapt our chosen solution of the inverse variational problem to re-examine and re-derive the expressions for nonstandard Lagrangians as written by Musielak [11] and by Cieśliński and Nikiciuk [12]. In each of the case studies presented the authors assume a form of the Lagrangian and then identify the equation that admits the chosen Lagrangian description. El-Nabulsi [13] introduced a class of nonstandard Lagrangians by working with new types of action functionals written as

$$S = \int_a^b e^{L(x, \dot{x}, t)} dt \tag{3}$$

and

$$S = \int_a^b L^{1-\gamma}(x, \dot{x}, t) dt. \tag{4}$$

In this case also expressions for nonstandard Lagrangians are chosen rather than derived from the solutions of inverse variational problems. We shall see that the solution of the inverse problem for Lagrangian in (3) and (4) is more complicated than that provided by Lopez [7].