EXISTENCE OF OPTIMAL CONTROLS FOR A SEMILINEAR COMPOSITE FRACTIONAL RELAXATION EQUATION*

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We consider a control system governed by a semilinear composite fractional relaxation equation in Banach space $\mathbb{X}$. Under suitable conditions on the data, we prove that the system has a mild solution. Then, we investigate the existence of an optimal state-control pair, solution to the associated Lagrange optimal control problem. An example is also given to illustrate our results.

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1. Introduction

In this paper, we consider a control system governed by the following semilinear composite fractional relaxation equation in a Banach space $\mathbb{X}$,

$$\begin{cases} x'(t) = AD^{\alpha}x(t) - x(t) + f(t, x(t)) + B(t)u(t), \quad 0 < t \le b, \\ x(0) = x_0, \\ u(t) \in U(t), \quad \text{a.e. } 0 \le t \le b, \end{cases} \tag{1}$$

where the state variable $x(\cdot)$ takes values in a Banach space $\mathbb{X}$, x' is the first derivative of x with respect to t, D^{α} is the Caputo fractional derivative of order $0 < \alpha < 1$. The operator $-A : D(A) \to \mathbb{X}$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{T(t), t \ge 0\}$ and $U(\cdot)$ is

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the control constraint multifunction in a Banach space $\mathbb{Y}$. The operator B belongs to $L^{\infty}([0,b],\mathscr{L}(\mathbb{Y},\mathbb{X}))$ where $\mathscr{L}(\mathbb{Y},\mathbb{X})$ is the space of bounded linear operators from $\mathbb{Y}$ into $\mathbb{X}$.

The main purpose of this paper is the following Lagrange optimal control problem:

$$\inf\{J(x,u),(x,u) \text{ subject to } (1)\}, \tag{2}$$

where J is a functional defined on $\mathbb{X}\times\mathbb{Y}$ by

$$J(x,u)=\int_0^b L(t,x(t),u(t))\,dt$$

and L is a given suitable integrand.

Problem (2) is a fractional optimal control problem. The theory of fractional differential equations has received much attention over the past twenty years, since they are important in describing the natural models such as diffusion processes, stochastic processes, finance and hydrology. We refer for instance to the books [16, 20], the recent papers [3, 10–14, 23, 24] and the references therein. Eq. (1) without the control term has already been considered by several authors. For instance, when the function f and the operator A are reduced to positive constants, Gorenflo and Mainardi [30] treated Eq. (1) using the Laplace transform method. In [10, 11], Lizama *et al.* investigated the existence and qualities properties of solutions of (1) by means of (a,k)- regularized resolvent. We also refer to [11, 31, 32] where the existence of solutions of a variation of Eq. (1):

$$x''(t)=AD^{\alpha}x(t)-x(t)+f(t,x(t)),\quad 0<t\le b,\quad 0<\alpha<2$$

is studied. On the other hand, the general literature on optimal controls for the differential evolution equations is extensive and different topics on optimal controls are considered. Concerning the calculus of variations and optimal control of fractional differential equations, some results are by now known, we refer the reader to the books [5, 9], to the recent papers [1, 2, 6, 15, 17–19, 21–27, 29] and the references therein. However, to the best of our knowledge, none of these works is concerned with the control of Eq. (1).

In this paper, we prove by means of probability densities the existence of a mild solution to the semilinear composite fractional Eq. (1). Then we show that under suitable assumptions on the integrand L, the Lagrange optimal control problem (2) has an optimal solution. Finally, we obtain the existence of optimal solutions for a concrete composite fractional relaxation equation concerned with the Basset problem by the theorems developed in this paper.

The paper is organized as follows. In Section 2, we give some basic definitions and results for fractional order integration and multivalued analysis. In Section 3, we prove the existence and uniqueness of mild solutions for (1). In Section 4, we establish the sufficient conditions to guarantee the existence of optimal controls for the Lagrange control problem (2).