REPRESENTATION OF CONCRETE LOGICS AND CONCRETE GENERALIZED ORTHOMODULAR POSETS

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In the present paper, we deal with the question when an effect algebra, resp. a generalized effect algebra, can be represented in the projection lattice of a Hilbert space. We show that such representability is closely related to the existence of a rich set of two-valued and Jauch–Piron states, resp. generalized two-valued and Jauch–Piron states.

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1. Introduction

Quantum logics were introduced in early thirties as the mathematical models of quantum events. Owing to the famous Heisenberg uncertainty relations, it was recognized that the classical rules of the Kolmogorov probability theory are not satisfied by quantum mechanical measurements. Therefore it was necessary to find a suitable generalization of Boolean algebras in order to describe quantum events. Quantum logics, or orthomodular sigma-lattices from the mathematical point of view, can be viewed as a most natural nondistributive abstraction of the set of projection operators on a Hilbert space, which is the basis of the traditional von Neumann approach to quantum mechanics.

Effect algebras (EAs) were introduced by Foulis and Bennett [4] in order to model unsharp quantum measurements, too. The prototype of effect algebras is the set of quantum effects, that is, self-adjoint operators between the zero and identity operator on a Hilbert space. Quantum effects play an important role in the mathematical description of quantum measurements, as the most general mathematical model of

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quantum observables are the normalized positive-operator-valued measures (POVMs) [2], which have their ranges in the set of quantum effects.

Effect algebras are a generalization of many algebraic structures which arise in mathematical physics, as well as other branches of mathematics, in particular Boolean algebras, orthomodular posets and lattices appearing in noncommutative measure theory, and also MV-algebras in fuzzy measure theory. An account of the axiomatic approach to quantum mechanics, employing EAs and the closely related D-posets [11], can be found in [3].

Several authors have encountered, studied, or employed algebraic structures that, roughly speaking, are EAs without a largest element, i.e. generalized effect algebras (GEAs). Going back to M. H. Stone's work [19] on generalized Boolean algebras, which was extended by M. F. Janowitz [9] to generalized orthomodular lattices, more recent works include: D. Foulis and M. Bennett [4] (positive cones in partially ordered abelian groups), J. Hedlíková and S. Pulmannová [8] (generalized orthoalgebras), G. Kalmbach and Z. Riečanova [10] (abelian RI-posets and abelian RI-semigroups), F. Kôpka and F. Chovanec [11] (D-posets), A. Mayet-Ippolito [12] (generalized orthomodular posets), M. Polakovič and Z. Riečanova [16] (densely defined positive operators) and A. Wilce [20] (cancellative positive partial abelian semigroups).

Increased interest in GEAs can be attributed to the discovery that certain systems of (possibly) unbounded positive symmetric operators on a Hilbert space, e.g. operators that represent quantum observables and states, can be organized into GEAs [17].

Recently, it has been shown in [18] that every effect algebra possessing an order determining set of states can be embedded into an effect algebra of quantum effects. In more detail, if E is an effect algebra and $\mathcal{S}$ is an order determining set of states on E, then there is an injective effect algebra morphism from E into the effect algebra of multiplication operators between the zero and identity operator on the complex Hilbert space $\ell_2(\mathcal{S})$.

In [14], the question of embedding of an MV-algebra into a set of quantum effects was investigated. It was shown that for every Archimedean MV-algebra, we can choose for the order determining set of states an order determining set of extremal states. This enables us to show that there is an injective MV-algebra morphism into the effect algebra of all multiplication operators between the zero and identity operator on $\ell_2(\mathcal{S})$ which, as a maximal set of commuting effects, is in fact an MV-algebra.

In [13], the question of representability of a generalized effect algebra in a generalized effect algebra of operators densely defined on a complex Hilbert space was studied. It was shown that a generalized effect algebra is representable in the operator generalized effect algebra $\mathcal{G}_D(\mathcal{H})$ if and only if it has an order determining set of generalized states.

In [6], Greechie has found an example of a finite orthomodular lattice with an order determining set of states which is not representable in the lattice $\mathcal{L}(\mathcal{H})$ of all closed subspaces (orthogonal projections, equivalently) of a separable complex Hilbert space.