AN OPERATOR APPROACH TO THE RATIONAL SOLUTIONS OF THE CLASSICAL YANG–BAXTER EQUATION

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(*Received March 26, 2009*)

Motivated by the study of the operator forms of the constant classical Yang–Baxter equation given by Semenov-Tian-Shansky, Kupershmidt and others, we try to construct the rational solutions of the classical Yang–Baxter equation with parameters by means of certain linear operators. The fact that the rational solutions of the CYBE for the simple complex Lie algebras can be interpreted in term of certain linear operators motivates us to give the notion of $\mathcal{O}$-operators such that these linear operators are the $\mathcal{O}$-operators associated to the adjoint representations. Such a study can be generalized to the Lie algebras with nondegenerate symmetric invariant bilinear forms. Furthermore we give a construction of a rational solution of CYBE from an $\mathcal{O}$-operator associated to the coadjoint representation and an arbitrary representation with a trivial product in the representation space, respectively.

2000 Mathematics Subject Classification: 81R, 17B

Keywords: Lie algebra, CYBE with parameters, Rational solutions.

1. Introduction

The classical Yang–Baxter equation (CYBE) first arose in the study of the inverse scattering theory (see [1, 2]) and has played an important role in the study of the classical integrable systems ([3–9] etc.). There are some close relations between it and many branches of mathematical physics and pure mathematics, like symplectic geometry, quantum groups, quantum field theory and so on (see [10] and the references therein).

The classical Yang–Baxter equation with spectral parameters is given as

$$[[r,r]] = [r_{12}(u_1,u_2), r_{13}(u_1,u_3)] + [r_{12}(u_1,u_2), r_{23}(u_2,u_3)]$$

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$$+ [r_{13}(u_1, u_3), r_{23}(u_2, u_3)] = 0, \tag{1.1}$$

where r is a function $r : \mathbb{F} \otimes \mathbb{F} \to \mathfrak{g} \otimes \mathfrak{g}$ with $\mathfrak{g}$ being a Lie algebra over a field $\mathbb{F}$ and r_{ij} is given as follows. For any $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$, set

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \qquad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \qquad r_{23} = \sum_i 1 \otimes a_i \otimes b_i, \tag{1.2}$$

and the commutation relations in (1.1) are given in the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$.

Most of the study on the classical Yang–Baxter equation (1.1) concentrates on the following cases ([11–14]): $\mathfrak{g}$ is taken as a finite-dimensional simple Lie algebra over the complex number field $\mathbb{C}$ and r is nondegenerate and depends on a single parameter. That is, r satisfies

$$r(u_1, u_2) = r(u_1 - u_2), \tag{1.3}$$

and there is no proper subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $r(u) \in \mathfrak{h} \otimes \mathfrak{h}$.

According to Belavin and Drinfeld ([11, 12]), the nondegenerate solutions of the classical Yang–Baxter equation (1.1) depending on a single parameter for the simple complex Lie algebras are divided into three cases: trigonometric, elliptic and rational. In this paper, we pay attention to the rational solutions r with exactly one pole. In fact, a general form of a rational solution r of CYBE can be written as [11–16]

$$r(u_1, u_2) = \frac{t}{u_1 - u_2} + r_0(u_1, u_2), \tag{1.4}$$

where t is the Casimir element of $\mathfrak{g}$ and r_0 is a polynomial in $\mathfrak{g}[u_1] \otimes \mathfrak{g}[u_2]$. However, it is not easy to get an explicit expression of r_0 from Eq. (1.4). Moreover, it is also difficult to extend the study from the simple complex Lie algebras to other Lie algebras.

On the other hand, for any $r \in \mathfrak{g} \otimes \mathfrak{g}$, r can be expressed by a matrix if we choose a basis. So it is natural to consider the conditions satisfied by the linear maps corresponding to the matrices (classical r-matrices) satisfying CYBE. For the constant solutions of CYBE, Semenov-Tian-Shansky [5] first gave an operator form of CYBE as a linear map $R : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]), \qquad \forall x, y \in \mathfrak{g}. \tag{1.5}$$

It is equivalent to the tensor form of the CYBE when the following two conditions are satisfied: (a) there exists a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ and (b) r is skew-symmetric. Note that Eq. (1.5) is exactly the Rota–Baxter relation of weight-zero in the version of Lie algebras [17–19], whereas the Rota–Baxter relations were introduced to generalize the integration-by-parts formula [20–22] and then (the versions of associative algebras) played important roles in many fields of mathematics and mathematical physics (cf. [23] and the references therein).

Furthermore, Kupershmidt [24] replaced the above condition (a) by letting r be a linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$ and when r is skew-symmetric, the tensor form of