THE RIGGED HILBERT SPACES APPROACH IN SINGULAR PERTURBATION THEORY

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We discuss a new approach in singular perturbation theory which is based on the method of rigged Hilbert spaces. Let A be a self-adjoint unbounded operator in a state space $\mathcal{H}_0$ and $\mathcal{H}_- \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_+$ be the rigged Hilbert space associated with A in the sense that $\mathrm{dom}A = \mathcal{H}_+$ in the graph-norm. We propose to define the perturbed operator $\tilde{A}$ as the self-adjoint operator uniquely associated with a new rigged Hilbert space $\tilde{\mathcal{H}}_- \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_+$ constructed using a given perturbation of A. We show that the well-known form-sum and self-adjoint extensions methods are included in the above construction. Moreover, we show that the super singular perturbations may also be described in our framework.

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1. Introduction

Let $A = A^* \geq 1$ be an unbounded self-adjoint operator in a Hilbert space $\mathcal{H}_0$ with the inner product $(\cdot, \cdot)_0$. And let

$$\mathcal{H}_- \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_+ \tag{1.1}$$

be the rigged Hilbert space associated with A in the sense that the domain $\mathrm{Dom}\, A = \mathcal{H}_+$ in the graph-norm. Here the symbol $\sqsupset$ means dense and continuous embedding. We note that a given pre-rigged pair $\mathcal{H}_0 \sqsupset \mathcal{H}_+$, the Hilbert space $\mathcal{H}_-$ is uniquely defined as the conjugate space to $\mathcal{H}_+$ with respect to $\mathcal{H}_0$ (for details see [8, 9]).

Besides the triplet (1.1) we will use also the chain of five spaces

$$\mathcal{H}_- \sqsupset \mathcal{H}_{-1} \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_1 \sqsupset \mathcal{H}_+, \tag{1.2}$$

where $\mathcal{H}_1 = \operatorname{Dom} A^{1/2}$, and $\mathcal{H}_{-1}$ is the completion of $\mathcal{H}_0$ in the norm $\|\cdot\|_{-1} = \|A^{-1/2}\cdot\|$.

Given $A = A^*$, another self-adjoint operator $\tilde{A}$ in $\mathcal{H}_0$ is said to be a purely singular perturbation of A if the set

$$\mathcal{D} := \{f \in \operatorname{Dom} A \cap \operatorname{Dom} \tilde{A} : Af = \tilde{A}f\} \text{ is dense in } \mathcal{H}_0 \tag{1.3}$$

(see [3, 5, 15–17, 20–30]). Under condition (1.3) we write $\tilde{A} \in \mathcal{P}_s(A)$ if $\tilde{A}$ is bounded from below. We write $\tilde{A} \in \mathcal{P}_{ws}(A)$ if $\operatorname{Dom} A^{1/2} = \operatorname{Dom} \tilde{A}^{1/2}$ (ws means weakly singular, i.e. a perturbation belongs to the $\mathcal{H}_{-1}$-class), and $\tilde{A} \in \mathcal{P}_{ss}(A)$ if the set $\mathcal{D}$ is dense in $\mathcal{H}_1$ (ss stands for strongly singular, i.e. a perturbation belongs to the $\mathcal{H}_{-2}$-class). Thus $\mathcal{P}_s(A) \supset \mathcal{P}_{ws}(A) \cup \mathcal{P}_{ss}(A)$.

It is clear that for each $\tilde{A} \in \mathcal{P}_s(A)$ there exists a densely defined symmetric operator

$$\mathring{A} := A|\mathcal{D} = \tilde{A}|\mathcal{D}$$

with nontrivial deficiency indices $\mathbf{n}^{\pm}(\mathring{A}) = \dim \ker(\mathring{A} \mp z)^* \neq 0$, $\operatorname{Im} z \neq 0$. Therefore each $\tilde{A} \in \mathcal{P}_s(A)$ may be defined as a self-adjoint extension of $\mathring{A}$, different from A. In singular perturbation theory each $\tilde{A}$ is fixed by some abstract boundary condition, which corresponds to a singular perturbation. In turn a singular perturbation is usually presented by a singular quadratic form γ given in the rigged Hilbert space (1.1).

In the present paper we propose to use a singular quadratic form γ (corresponding to a perturbation) for the construction of a new chain of Hilbert spaces similar to (1.2),

$$\tilde{\mathcal{H}}_- \sqsupset \tilde{\mathcal{H}}_{-1} \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_1 \sqsupset \tilde{\mathcal{H}}_+, \tag{1.4}$$

and then to define the perturbed operator $\tilde{A}$ as an operator associated with this new rigging (1.4).

In the paper, see below Theorem 5.1, Theorem 5.2, Theorem 6.1, and Theorem 7.1 we establish a one-to-one correspondence between three families of objects: singular perturbations $\tilde{A} \in \mathcal{P}_{ss}(A)$, rigged Hilbert spaces of the form (1.4), and singular quadratic forms γ with fixed properties. We extend this one-to-one correspondences to a more general set of objects involving super singular perturbations.

2. Singular quadratic forms in A-scales

Let $A \geq 1$ be a self-adjoint unbounded operator in a separable Hilbert space $\mathcal{H}_0$ which is equipped in such a way that the domain $\operatorname{Dom} A = \mathcal{H}_+$ in the norm $\|\cdot\|_+ := \|A\cdot\|$ (see (1.1)).

In the paper we discuss a new construction of singularly perturbed operator $\tilde{A}$ in $\mathcal{H}_0$. Namely, we define $\tilde{A}$ as the operator associated with a new rigged Hilbert space $\tilde{\mathcal{H}}_- \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_+$, where $\tilde{\mathcal{H}}_+ = \mathcal{D}(\tilde{A})$. The inner product $(\cdot,\cdot)_+^{\sim}$ in $\tilde{\mathcal{H}}_+$ is defined as a perturbation of the inner product $(\cdot,\cdot)_+$ in $\mathcal{H}_+$. Formally one can write $(\cdot,\cdot)_+^{\sim} = (\cdot,\cdot)_+ + \gamma(\cdot,\cdot)$, where the form γ corresponds to a singular perturbation.

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