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# Global existence and blow up of solutions for Cauchy problem of generalized Boussinesq equation

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**Abstract**

We study the Cauchy problem of generalized Boussinesq equation $u_{tt} - u_{xx} + (u_{xx} + f(u))_{xx} = 0$, where $f(u) = \pm|u|^p$ or $\pm|u|^{p-1}u$, $p > 1$. By introducing a family of potential wells we obtain invariant sets, vacuum isolating and threshold result of global existence and nonexistence of solution.


*Keywords:* Generalized Boussinesq equation; Cauchy problem; Potential wells; Global existence; Blow up

## 1. Introduction

It is well known that Boussinesq equation

$$u_{tt} - u_{xx} + (u_{xx} \mp u^2)_{xx} = 0 \tag{1.1}$$

is a very important and famous high order strongly nonlinear mathematical physics model equation, which is introduced to describe the motion of water wave with small-amplitude long waves. The Cauchy problem of the generalized Boussinesq equation

$$u_{tt} - u_{xx} + (u_{xx} + f(u))_{xx} = 0, \tag{1.2}$$

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R} \tag{1.3}$$

has been studied for a long time and there has been a lot of results [1–19]. However, on the global existence, nonexistence and finite time blow up of solution for problems (1.2) and (1.3), there are only a few results mainly in [1–4]. In [1] Bona and Sachs studied the problem for $f(u) = |u|^{p-1}u$, $p > 1$. By using Kato's abstract theory of quasilinear evolution equation, they proved the existence of local $H^{s+2} \times H^s$ solution for any $(u_0, u_1) \in H^{s+2} \times H^s$ with $s > \frac{1}{2}$. For $1 < p < 5$ they proved the global existence of $H^{s+2} \times H^s$ solution under some assumptions on initial data. In [2] Linares established the local well posedness of problems (1.2) and (1.3) with $f(u) = |u|^\alpha u$ for $H^1 \times L^2$ solution when $0 < \alpha < \infty$ and for $L^2 \times H^{-1}$ solution when $0 < \alpha < 4$ respectively. He also proved that these local solutions can be extended globally when the size of the data is small. In [3] Liu Yue studied problems (1.2) and (1.3) for $f(u) = |u|^{p-1}u$ also. By using potential well method he obtained invariant sets of solutions and proved the global existence and finite time blow up of solutions. Also, a threshold result of global existence and nonexistence of solution was given. In [4] Ruying Xue studied problems (1.2) and (1.3) for $f(u) = u^{k+1}$, $k = 1, 2, \ldots$. For $k > 4$ and small initial data global existence of solution was proved in Besov space.

Note that in all of the above studies except [3] only the local and global existence of solution was proved, but the global nonexistence and finite time blow up of solution for problems (1.2) and (1.3) were not discussed yet. Certainly any threshold result of global existence or nonexistence of solution was not obtained either. In [3], although all of the above problems were considered, the obtained results are only for a special

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case $f(u) = |u|^{p-1}u$. In addition, in [1–3] only the case $f(u) = |u|^{p-1}u$ was considered, which does not include the cases $f(u) = \pm u^2$ and their generalization $f(u) = \pm|u|^p$, $p > 1$. Except the following two cases: (i) $f(u) = u^{2k}$, $k \geq 3$ in Besov space in [4] and (ii) $f(u) = u^2$ in [2], the results in the above literatures can not include the cases $f(u) = \pm u^2$ and their generalization $f(u) = \pm|u|^p$, $p > 1$. Based on the facts above, we cannot make sure that the results for $f(u) = |u|^{p-1}u$ can be applicable for the cases $f(u) = \pm|u|^p$. Therefore for problems (1.2) and (1.3) the following problems are still open up to now.

(i) The global existence and nonexistence of solution for the case $f(u) = \pm|u|^p$ or $-|u|^{p-1}u$, $p > 1$.
(ii) For the cases $f(u) = \pm|u|^p$, $p > 1$, whether there exists a threshold result of global existence and nonexistence of solution.
(iii) For the case $f(u) = \pm|u|^p$, $p > 1$, whether there exist some invariant sets of solutions.
(iv) The global existence of solution for problems (1.2) and (1.3) with critical initial condition $E(0) = d$, where $E(0)$ is the initial energy and $d$ is the depth of the potential well defined for problems (1.2) and (1.3).

Recently, in [20] the first author of this paper introduced a family of potential wells for the initial boundary value problem of semilinear wave equation

$$u_{tt} - \Delta u = |u|^{p-1}u.$$

By means of the family of potential wells, the vacuum isolating of solutions and some new results on global existence and invariant sets of solutions were obtained. The purpose of this paper is to make the above problems clear by introducing the family of potential wells. In this paper we introduce a family of potential wells $\{W_\delta\}$ and the corresponding family of outside sets $\{V_\delta\}$. Then by using them we not only obtain the invariant sets and vacuum isolating of solutions, but also give some threshold results of global existence and nonexistence of weak solutions and smooth solutions. Finally, we prove the global existence or nonexistence of solution for problems (1.2) and (1.3) with critical initial condition $E(0) = d$. So all of the above open problems are resolved.

Throughout the present paper, the following notations are used for precise statements: $L^p$ denotes $L^p(\mathbb{R})$, $H^s$ denotes $H^s(\mathbb{R})$ and $\|u\|_p = \|u\|_{L^p(\mathbb{R})}$, $\|u\|_{H^s} = \|u\|_{H^s(\mathbb{R})}$, $\|u\| = \|u\|_{L^2(\mathbb{R})}$; and the inner product $(u, v) = \int_\Omega uv\mathrm{d}x$, $(u, v)_{H^1} = (u, v) + (u_x, v_x)$.

In [1,3] in order to prove the local existence of solution, the problems (1.2) and (1.3) is reduced to an equivalent system of equations

$$\begin{cases} u_t = v_x, \\ v_t = (u - u_{xx} - f(u))_x, \end{cases} \tag{1.4}$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x). \tag{1.5}$$

Note that for $p > 1$, both $|s|^p$ and $|s|^{p-1}s$ belong to $C^1(R)$. Hence from [3] we can obtain the following local existence theorem of weak solution.

**Propostion 1.1.** *Let $f(u) = \pm|u|^p$ or $\pm|u|^{p-1}u$, $p > 1$, $(u_0, v_0) \in H^1 \times L^2$. Then problems (1.4) and (1.5) admits a unique local weak solution $(u, v) \in C\left([0, T); H^1 \times L^2\right)$. Moreover if $T_m$ is the maximal existence time of $(u, v)$, then there holds either*

(i) $T_m = +\infty$,
*or*
(ii) $T_m < \infty$ *and* $\limsup_{t\to T_m}\left(\|u\|_{H^1} + \|v\|\right) = +\infty$.

From $u_t = v_x$ we can obtain $\|v\| = \|\wedge^{-1} u_t\|$, where $\wedge^{-\alpha}\varphi = \mathscr{F}^{-1}\left(|\xi|^{-\alpha}\mathscr{F}(\varphi)\right)$, $\mathscr{F}$ and $\mathscr{F}^{-1}$ denote the Fourier transformation and inverse Fourier transformation respectively. Hence from Proposition 1.1 we can obtain the following

**Corollary 1.2.** *Let $f(u) = \pm|u|^p$ or $\pm|u|^{p-1}u$, $p > 1$, $u_0 \in H^1$ and $\wedge^{-1} u_1 \in L^2$. Then problems (1.2) and (1.3) admits a unique local weak solution $u \in C\left([0, T); H^1\right)$ with $\wedge^{-1} u_t \in C\left([0, T); L^2\right)$. Moreover if $T_m$ is the maximal existence time of $u$, then there holds either*

(i) $T_m = +\infty$,
*or*
(ii) $T_m < \infty$ *and* $\limsup_{t\to T_m}\left(\|u\|_{H^1}^2 + \|\wedge^{-1} u_t\|^2\right) = +\infty$.

From Theorems 3 and 4 in [1] we can obtain the following

**Propostion 1.3.** *Let $f(u) = \pm u^{k+1}$, $k = 1, 2, \ldots$, $u_0 \in H^{s+2}$ and $u_1 \in H^s$ for some $s > \frac{1}{2}$. Then problems (1.2) and (1.3) admit a unique local smooth solution $u(t) \in C\left([0, T); H^{s+2}\right) \cap C^1\left([0, T); H^s\right) \cap C^2\left([0, T); H^{s-2}\right)$. Moreover if $T_m$ is the maximal existence time of $u(t)$ and there exists a constant $M_0$ such that*

$$\|u(t)\|_{H^1} + \|v(t)\| \leq M_0 \text{ for all } t \in [0, T_m).$$

*Then $T_m = +\infty$.*

**Corollary 1.4.** *Under the condition of Proposition 1.3 $T_m < \infty$ if and only if*

$$\limsup_{t\to T_m}\left(\|u\|_{H^1}^2 + \|\wedge^{-1} u_t\|^2\right) = +\infty.$$

## 2. Introducing of families $\{W_\delta\}$ and $\{V_\delta\}$

In this section we first give some preliminary lemmas, then by using them we introduce two families $\{W_\delta\}$ and $\{V_\delta\}$.

**Lemma 2.1.** *Let $f(u) = \pm|u|^p$ or $|u|^{p-1}u$, $p > 1$ and $F(u) = \int_0^u f(s)\mathrm{d}s$. Then*

(i) $uf(u) > 0$, $F(u) > 0$ *for* $u \neq 0$ *if* $f(u) = |u|^{p-1}u$;
$uf(u) > 0$, $F(u) > 0$ *for* $u > 0$ *if* $f(u) = |u|^p$;
$uf(u) > 0$, $F(u) > 0$ *for* $u < 0$ *if* $f(u) = -|u|^p$.
(ii) $f(\lambda u) = \lambda^p f(u)$, $F(\lambda u) = \lambda^{p+1}F(u)$, $\forall u \in \mathbb{R}, \lambda > 0$.
(iii) $|uf(u)| = |u|^{p+1}$, $|F(u)| = \frac{1}{p+1}|u|^{p+1}$, $\forall u \in \mathbb{R}$.
(iv) $(p + 1)F(u) = uf(u)$, $\forall u \in \mathbb{R}$.