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Phase compactons

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Abstract

We study the phase dynamics of a chain of autonomous, self-sustained, dispersively coupled oscillators. In the quasicontinuum limit the basic discrete model reduces to a Korteveg–de Vries-like equation, but with a *nonlinear dispersion*. The system supports compactons – solitary waves with a compact support – and kovatons – compact formations of glued together kink–antikink pairs that propagate with a unique speed, but may assume an arbitrary width. We demonstrate that lattice solitary waves, though not exactly compact, have tails which decay at a superexponential rate. They are robust and collide nearly elastically and together with wave sources are the building blocks of the dynamics that emerges from typical initial conditions. In finite lattices, after a long time, the dynamics becomes chaotic. Numerical studies of the complex Ginzburg–Landau lattice show that the non-dispersive coupling causes a damping and deceleration, or growth and acceleration, of compactons. A simple perturbation method is applied to study these effects.

Keywords: Lattice of nonlinear oscillators; Phase dynamics; Compacton

1. Introduction

The subject matter of this paper unifies two principal fields of nonlinear science: coupled self-sustained oscillators and soliton theory. Coupled autonomous oscillators have been a subject of interest since the discovery of their synchronization by Huygens [1]. A theoretical understanding of this phenomenon is almost one hundred years old [2]; since then different features of coupled oscillators have attracted considerable attention (see, e.g., [3,4]). When the coupling of periodic self-sustained oscillators is weak it can be described in the phase approximation [5], where only a variation of oscillator phases enters into play. For two coupled oscillators this leads to an Adler-type equation [6]. The corresponding phase models are widely used for a description of oscillator lattices [7–11] and globally coupled ensembles [5,12–15].

The phase approximation for coupled oscillators requires the coupling strength to be small compared to the smallest, in the absolute sense, negative Lyapunov exponent. One may then consider the 'amplitude' perturbations as slaved entities. In the absence of coupling the resulting phase equations have only zero Lyapunov exponents, therefore *the dissipative or conservative nature of the phase dynamics will solely depend on the particulars of the coupling*. In studies which focus on synchronization properties of oscillators, it is natural to assume that the coupling is dissipative which thus tends to equalize the phases. Adequately strong coupling then leads to a synchronous state with a uniform phase of a lattice or a network, if the coupling is attractive, or to an anti-phase lattice, if the coupling is repulsive. Notably, certain types of coupling lead to a conservative dynamics. A prominent example being that of a splay state in a globally coupled ensemble of oscillators [16–20].

In contradistinction to previous studies, in the present work we consider the dynamics of a one-dimensional lattice, a chain, of oscillators with a dispersive coupling. A multicore fiber laser [21], where individual self-oscillating lasers are arranged in a ring, may be a realization of such a lattice. Another physical example, an array of Josephson junctions, will be discussed below. Since both the local phase dynamics and the coupling are non-dissipative, such a system shares many

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properties with Hamiltonian lattices, in particular the phase volume is conserved. This means that if stable synchronized states are admissible, they are not attractors, and the dynamics is expected to be similar to that of the well-known Hamiltonian examples, like the sine-Gordon lattice, for which the basic building blocks are traveling solitary waves like pulses or kinks, that on integrable lattices collide elastically (see, e.g., [22,23]), while in non-integrable cases eventually give way to chaos.

In recent years two new concepts have significantly enlarged our understanding of nonlinear processes in Hamiltonian lattices and fields. One concept introduces localized periodic breathers in lattices [24]. The other introduces excitations in genuinely nonlinear lattices and wave equations. Unlike the usual solitons that have exponential, or algebraic, tails, the corresponding traveling waves have compact or almost compact support. These waves, the **compactons**, have been introduced by one of us [25,26] and put forward in [27–35]. Typically, compacton-bearing PDE equations (or spatially discrete equations on a lattice) are non-integrable, at least in the conventional sense, yet their remarkable robustness seems to have very little to do with the conventional solitonic integrability and appears to originate in the nonlinear mechanism which induces their compactness. Many of the underlying equations of motion do not have an energy integral and some may, under certain conditions, generate exploding solutions. Nevertheless, typical numerical simulations show that an initial perturbation of a finite span decomposes into a set of compactons. As an example we mention a recent modeling of DNA opening with one-dimensional Hamiltonian lattices [36]. Other examples include a compression wave in a granular chain [37–41] and sedimentation of particles in dilute suspensions [42].

In the present paper we study compactons in a chain of dispersively coupled nonlinear self-sustained oscillators (a short report was presented in [43]). In Section 2 we derive the basic model of dispersively coupled phase equations. In particular, we show that such a model emerges naturally in a chain of Ginzburg–Landau oscillators. Some general features of our model are presented in Section 3 where we derive in the quasicontinuum approximation a genuinely nonlinear PDE to describe the dynamics on a lattice which for small amplitudes reduces to the K(2, 2)-model for compactons [26]. In Section 4 we present the solitary solutions of the derived PDE and show that there are two types of compact waves: the usual compactons (solitary waves with a compact support) and kovatons (flat-top compactons or glued compact kinks). The corresponding solitary traveling solutions on the lattice are found numerically using an iterative algorithm due to Petviashvili. We show that *the exactly compact front is replaced with a superexponential tail where the discrete effects are essential*. This effect is confined to a very thin boundary layer which shrinks to a singular point in the quasicontinuum limit. In Section 5 we present numerical simulations of the dynamics on the lattice: evolution of an initial pulse, collisions of compactons and kovatons and other types of waves. In Section 6 we consider finite lattices and demonstrate the emergence of a spatio-temporal chaos of Hamiltonian type. In Section 7 we step beyond the phase approximation and show that compactons and kovatons can be also observed in the Ginzburg–Landau lattice. Here, however, additional small dissipative terms arise and lead to the decay, or growth, of compactons; these effects are addressed using a perturbation method.

2. The basic model

2.1. Phase lattice and variety of couplings

An autonomous periodic self-sustained oscillator with frequency ω can be characterized by the phase φ that obeys $\frac{d\varphi}{dt} = \omega$. An equation for weakly coupled oscillators may be derived in two steps (see [4,5] for details). First, one uses a smallness of the coupling compared to the smallest, in the absolute sense, negative Lyapunov exponent of the oscillator. This allows us to write equations for the phase evolution on a perturbed limit cycle. For the lattice of identical oscillators these equations read

$$\frac{d\varphi_n}{dt} = \omega + \tilde{q}(\varphi_{n-1}, \varphi_n) + \tilde{q}(\varphi_{n+1}, \varphi_n). \tag{1}$$

Here $\tilde{q}$ is a coupling function 2π-periodic in each argument. In the second step the smallness of the coupling compared to the frequency ω is used to average the r.h.s. of (1). Then only the 'slow' part of $\tilde{q}$ remains and is a function of phase differences:

$$\frac{d\varphi_n}{dt} = \omega + q(\varphi_{n-1} - \varphi_n) + q(\varphi_{n+1} - \varphi_n), \tag{2}$$

where $q(\varphi + 2\pi) = q(\varphi)$. Introducing new variables

$$v_n = \varphi_{n+1} - \varphi_n, \tag{3}$$

we rewrite the phase equations as

$$\frac{dv_n}{dt} = q(-v_n) + q(v_{n+1}) - q(-v_{n-1}) - q(v_n). \tag{4}$$

Since the frequency does not appear in (4), rescaling the time we may consider the coupling function q to be of order one.

Since in general any function q can be represented as a sum of its odd and even parts, we write q as $q(v) = q^o(v) + q^e(v)$ to obtain

$$\begin{aligned}\frac{dv_n}{dt} &= q^e(v_{n+1}) - q^e(v_{n-1}) + q^o(v_{n+1}) \\ &\quad + q^o(v_{n-1}) - 2q^o(v_n) \\ &= \nabla_d q^e(v) + \Delta_d q^o(v),\end{aligned} \tag{5}$$

where Δ_d and ∇_d are the discrete Laplacian and nabla operators, respectively:

$$\Delta_d f = f_{n+1} + f_{n-1} - 2f_n, \quad \nabla_d f = f_{n+1} - f_{n-1}. \tag{6}$$

Typical and probably the simplest choice for the coupling is $q(\varphi) = \sin\varphi$. This odd coupling is dissipative and leads to the system $\dot{v}_n = \Delta_d \sin(v)$ that has the synchronous state $v_n = 0$ as an attractor. We, on the other hand, shall restrict our attention to a purely even coupling function, yielding

$$\frac{dv_n}{dt} = q(v_{n+1}) - q(v_{n-1}) = \nabla_d q(v) \tag{7}$$