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Intrinsic transfer matrix method and split quaternion formalism for multilayer media

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HIGHLIGHTS

- Split quaternion formalism for 1D wave propagation in a multilayer medium is obtained based on the transfer matrix method.
- Periodic media with a defect are considered and several particular cases are analysed.
- A class of commutative split quaternions is identified, corresponding to defects that can be placed anywhere in the structure with the same effect.
- A medium composed of commutative split quaternion elements is described.

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ABSTRACT

The intrinsic transfer matrix method for 1D longitudinal elastic wave propagation through multilayer media is used to obtain an equivalent split quaternion formalism. Periodic media are analysed in this framework and the presence of a defect is considered. A simple one layer defect and an inversion defect are analysed. A commutative type of split quaternion is identified, which corresponds to defects of the periodic structure that can be placed in any position, the overall acoustic properties of the medium being conserved. Also, a nonperiodic medium composed of such commutative elements has a behaviour independent of the order of elements. Several possible applications in sound wave measurement and processing are outlined. The proposed split quaternion formalism is compact and can make analytical and numerical computations easier.

1. Introduction

Multilayer structures for elastic wave propagation are an interesting type of composite materials both theoretically and for their applications [1–3]. They have properties that extend and complement those of bulk materials. Periodic multilayer structures called sonic (phononic) crystals present a rich acoustic band structure in terms of frequency and have applications in sound insulation, environmental noise control or in filtering and sensing [4–6].

For applications, the bandgap positions and widths are important and they depend on the material properties, shape and size of layers and on topological structure. There are reported 1D, 2D and 3D periodic systems for which very large bandgaps are obtained [4,7,8]. A proper choice of materials gives the possibility of obtaining mechanically or electrically tunable phononic composites required in practice [9–11]. Band structures can be obtained computationally by different methods, such as: Bragg scattering and heuristic models [12], stochastic methods [6,13], plane wave expansion method

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(PWE) [14,15], finite difference time domain method (FDTD) [16–18], or the transfer matrix method (TM) [19–23]. In the 1D propagation case exact theoretical results have been obtained for a periodic structure [24] and in the presence of defects [21].

Here we describe a split quaternion formalism equivalent to the transfer matrix formalism [25–28] for wave propagation through 1D multilayered structures. A periodic structure with a defect is taken into consideration and the elements of the transfer matrix are obtained. Several particular cases are described. The use of split quaternions allows the identification of a class of commutative quaternions, which correspond to elements of the multilayer medium that can be placed anywhere inside its structure without changing the medium's properties. The formalism presented allows a simple and straightforward description of wave propagation through a multilayer medium, which is useful for both theoretical and computational approaches.

2. Transfer matrix method in Fourier space

We discuss the 1D sound wave propagation through an inhomogeneous multilayered rod, where the total wave is expressed as the superposition of a progressive wave and a regressive wave, both given by the Fourier transforms of the corresponding time-dependent waves. A matrix formalism describes the waves' space evolution as a function of the layers' properties.

Consider a rod composed of n layers made of different materials set along the rod's length, indexed $j = 1, 2, \ldots, n$. The rod has free input (in) and output (out) ends. The layers have the same transversal surface. Each layer j is characterised by: length l_j, mass density ρ_j, sound velocity c_j, and acoustic impedance $Z_j = \rho_j c_j$. For acoustic frequency f the wave number through layer j is $k_j = \omega/c_j$, where $\omega = 2\pi f$ is the angular frequency. Label $\mathrm{i} = \sqrt{-1}$. The Fourier transforms of the progressive and regressive waves $u_j^p(t)$, $u_j^r(t)$ at the output of layer j are taken:

$$A_j(\omega) = \int_{-\infty}^{\infty} u_j^p(t)\, e^{\mathrm{i}\omega t} dt, \qquad B_j(\omega) = \int_{-\infty}^{\infty} u_j^r(t)\, e^{\mathrm{i}\omega t} dt. \tag{1}$$

The propagation of the total wave through the rod from input to output can be modelled with a 2×2 intrinsic transfer matrix $\mathbf{T}$:

$$\begin{bmatrix} A_{\text{out}} \\ B_{\text{out}} \end{bmatrix} = \mathbf{T} \cdot \begin{bmatrix} A_{\text{in}} \\ B_{\text{in}} \end{bmatrix}. \tag{2}$$

The transfer matrix $\mathbf{T}$ can be obtained from the solution of the 1D wave equation and the continuity conditions of displacement and force at the boundary between two layers. Particular transfer matrices are [1,20,25–28] the propagation matrix $\mathbf{P}_j$ through a layer j and the discontinuity matrix $\mathbf{D}_{j,j+1}$ from layer j to layer $j+1$ at their interfaces:

$$\mathbf{P}_j = \begin{bmatrix} \exp(\mathrm{i}k_j l_j) & 0 \\ 0 & \exp(-\mathrm{i}k_j l_j) \end{bmatrix}, \qquad \mathbf{D}_{j,j+1} = \frac{1}{2} \begin{bmatrix} 1 + z_{j,j+1} & 1 - z_{j,j+1} \\ 1 - z_{j,j+1} & 1 + z_{j,j+1} \end{bmatrix}. \tag{3}$$

Here $z_{j,j+1} = Z_j/Z_{j+1}$ is the acoustic impedance of layer j relative to layer $j+1$. The intrinsic transfer matrix of the multilayer medium is $\mathbf{T} = \mathbf{P}_n \mathbf{D}_{n-1,n} \ldots \mathbf{D}_{23} \mathbf{P}_2 \mathbf{D}_{12} \mathbf{P}_1$. From the structure of the propagation and discontinuity matrices it follows that the transfer matrix has a general expression:

$$\mathbf{T} = \begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix}, \tag{4}$$

where a, b are complex quantities depending on the wave frequency and layers' properties and a^*, b^* are their conjugates. Its determinant is $\det \mathbf{T} = Z_1/Z_n = z_{1,n} = z$, *i.e.* the acoustic impedance of the input layer relative to the output layer.

3. Split quaternion formalism for wave propagation

A more detailed description of split quaternions [29–31] is presented in Appendix. Split quaternions can be expressed as pairs of complex numbers (the temporal and spatial components of the quaternion) and have a corresponding matrix representation. Unlike the complex number product, the split quaternion product is noncommutative. The split quaternion matrix representation can be used to compute the propagation matrix of a multilayered medium. Since propagation and discontinuity matrices correspond to timelike split quaternions, transfer matrices translate into timelike split quaternions with timelike, lightlike or spacelike vector parts.

Consider a multilayered element (the spatial period) with transfer matrix $\boldsymbol{\tau}$ that is repeated a certain number of times, and a "defective" multilayered element with transfer matrix $\boldsymbol{\tau}_d$:

$$\boldsymbol{\tau} = \begin{bmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{bmatrix}, \qquad \boldsymbol{\tau}_d = \begin{bmatrix} \alpha_d & \beta_d \\ \beta_d^* & \alpha_d^* \end{bmatrix}. \tag{5}$$