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Bloch theorem with revised boundary conditions applied to glide and screw symmetric, quasi-one-dimensional structures

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HIGHLIGHTS

- Wave propagation in structures with translational, glide and screw symmetries.
- Developed boundary conditions in Cartesian coordinates.
- The method is applicable to determine to complex wavenumber given a real frequency.
- The method is applicable to determine a real frequency given a real wavenumber.

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ABSTRACT

Bloch theorem is useful for analyzing wave propagation in periodic systems. It has been widely used to determine the energy bands of various translationally-periodic crystals and with the advent of nanoscale structures like nanotubes, it has been extended to account for additional symmetries using group theory. However, this extension is restricted to Hamiltonian systems with analytical potentials. For complex problems, as for engineering structures, the periodic unit cells are often discretized and the Bloch method is restricted to translational periodicity.

The goal of this paper is to generalize the direct and transfer-matrix propagation Bloch method to structures with glide and screw symmetries by deriving appropriate boundary conditions. Dispersion relations for a set of reduced problems are compared to results from the classical method, when available. It is found that (i) the dispersion curves are easier to interpret, (ii) the computational cost and error are reduced, and (iii) revisited Bloch method is applicable to structures as the Boerdijk–Coxeter helix that do not possess purely-translational symmetries for which the classical method is not applicable.

1. Introduction

Wave propagation in structures is of interest in a large range of applications such as non-destructive evaluation for structural health monitoring [1] and imaging [2]. In the case of translationally-periodic structures, Bloch theorem, the extension of Floquet-theory to three-dimensions, is used to obtain the behavior of an infinite medium from the analysis of a single unit cell [3]. This can be used for example to compute the electronic band-gap structures in crystals [4] as well as dispersion relations of railways [5]. However, in the presence of symmetries other than translation, Bloch theorem in its original form cannot be used.

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These challenges are being addressed with two approaches: in modern physics, Hamiltonian systems with analytical or simple potentials can be analyzed via group theory [6], and find applications among nanotubes, nanoribbons, DNA, proteins or polymers structures [6–10]. In [11], Crepeau has shown that for electromagnetic waves propagating in a glide or screw-symmetric antenna, the introduction of special operators for the boundary conditions allows a reduction of the unit cell. This work is articulated in [12] where electromagnetic slow-waves in helical structures are investigated. The consideration of symmetries allows significant simplifications and reduces computation cost.

For structural wave propagation, the presence of symmetries is handled by periodic-boundary conditions. For translation symmetries, the employment of Bloch theorem is easily implemented in finite elements (FE), and significant research has been devoted to analyzing wave propagation in engineering structures. One of the early pioneers was Mead [13] who studied wave propagating in railways, restricting the analytical analysis to a periodic, simply-supported beam. He introduced the transfer-matrix approach which relates the displacements and the forces on both sides of the periodic element for a given frequency. Once coupled to the propagation constant μ (dimensionless wave number), a concept introduced by Heckl [14] linking also both side forces and displacements, the system can be recast into an eigenvalue problem for which e^{μ} are the eigenvalues. The transfer-matrix approach has been then adapted to waveguides FE (WFE), introducing the ability to analyze vibration in complex finite [15] and infinite [16,17] periodic structures. Initially available for one-dimensional problems, this technique has been extended to two-dimensional problems for portions of the irreducible Brillouin zone (IBZ) [18], and more recently for the entire IBZ [19,20]. However, for complex periodic unit cells with a large number of degrees of freedom (dof), the transfer matrix may be ill-conditioned leading to spurious eigenvalues and a better alternative consists of recasting the system into another eigenvalue problem for which $\cosh^{-1}(\mu)$ are the eigenvalues [21]. An alternative technique to the transfer-matrix approach assumes harmonic waves (propagating without attenuation) and reduces the system to an eigenvalue problem for which, the frequencies are the solutions. This method, referred to as direct, can easily be implemented in existing FE software [22], and it is conveniently used to evaluate the wave response of periodic media with microstructure [23–28]. These methods however cannot immediately be extended to systems with symmetries other than translation.

In this paper, appropriate boundary conditions are approximated for glide (also called glide-plane) and screw (also called screw-axis) symmetric systems for both the direct and the transfer-matrix approaches. It follows previous work on the dispersion of a periodic buckled beam [29], a structure which is glide-symmetric. In [30–32], glide-symmetric Warren trusses and undulated beams are considered without mentioning symmetries, while dispersion curves fold, a characteristic due to the fact that the selected period is not the minimum one, as it will be shown in the present paper. In [33], vibrations of a tire are obtained thanks to the WFE method restricting analyses to a small portion of the circumference using cyclic symmetry. In [34], Stephen studies the vibration of finite pre-twisted periodic structures taking advantage of screw-symmetry and uses a rotation matrix to link consecutive unit cells.

In the literature pertaining to wave propagation in infinite periodic structures, helical waveguides with constant cross-sections as cables have been addressed taking advantage of the symmetry. At least two alternatives exist for the choice of the unit cell: modeling a three-dimensional slice of the waveguide using classical FE [35], or restricting the modeling to its cross-section using semi-analytical finite element (SAFE) [36,37], gaining in computation cost but requiring the implementation of special FE formulations. The proposed method differs in the choice of the reference coordinate system. Indeed, the formulation of [35–37] is derived in the helical coordinate system, whereas the Cartesian coordinate system is used providing a simplified formulation and a natural choice for FE. Moreover, in the present paper, no assumption on constant cross-section is made, such that the method is generalized to the full screw-symmetric group.

This paper completes the initial work [38] and is organized as follows. In Section 2, appropriate boundary conditions are proposed which reduce unit-cell based on translation symmetry to simplified subcells according to additional symmetries. In Appendix, it is demonstrated that these boundary conditions leave the eigenvalues of the dynamic equation unchanged, in accordance with the present symmetries for both the transfer-matrix approach and the direct one. In Section 3 the applicability of the proposed method is demonstrated in a set of quasi-one dimensional problems, with respect to wave propagation. Conclusions follow.

2. Reduced Bloch theorem

Wave propagation in periodic structures can be investigated through the analysis of a unit cell and the application of Bloch theorem [3,5,17]. The motion of a linear, periodic domain resulting from uniaxial wave propagation may be expressed as follows:

$$\boldsymbol{d}_n = \boldsymbol{d}_0(\mu(\omega))e^{\mu n}, \tag{1}$$

where $\boldsymbol{d}_n$ denotes the displacement vector of cell n within the periodic assembly, and $\boldsymbol{d}_0$ is the displacement vector within the reference cell. The propagation constant, μ, is a complex number ($\mu = \mu_r + \mathrm{i}\mu_i$, $\mathrm{i} = \sqrt{-1}$) where the real and imaginary parts represent respectively the attenuation and phase constants. Given the periodicity, the propagation constant μ is equal to the wave number κ multiplied by the spatial period L such that $\mu = L\kappa$. The set of linear ordinary differential equations for a discretized periodic unit cell is:

$$\mathbf{M}\ddot{\boldsymbol{d}} + \mathbf{K}\boldsymbol{d} = \boldsymbol{f}, \tag{2}$$