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Diffraction of waves on square lattice by semi-infinite rigid constraint

Basant Lal Sharma

Department of Mechanical Engineering, Indian Institute of Technology Kanpur, Kanpur, U. P. 208016, India

HIGHLIGHTS

- Exact solution for diffraction of time-harmonic lattice wave by a semi-infinite defect.
- Far-field asymptotic approximation of the exact solution.
- Graphical comparison with numerical solution for a set of frequencies in the pass band.
- Low frequency limit of the exact solution to obtain continuous integral form.
- Relevance to five point discretization of the two-dimensional Helmholtz equation.

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ABSTRACT

The problem of diffraction of a time harmonic lattice wave in a two-dimensional square lattice, by a semi-infinite rigid constraint, is investigated as a discrete analogue of diffraction by a Sommerfeld 'soft' half plane. The discrete Helmholtz equation, with input data prescribed on a semi-infinite row of lattice sites, is solved exactly using the discrete Wiener–Hopf method. The far-field asymptotic approximation of exact solution is provided. The scattered wave, in far field, is compared with a numerical solution of the problem for a set of frequencies in the pass band. The low frequency approximation of the exact solution is derived and it coincides with the Sommerfeld's solution in its integral form. The results and discussion associated with the discrete Sommerfeld problem are relevant to numerical methods based on a 5-point discretization of the two-dimensional Helmholtz equation. In addition to the mechanics of waves in lattices, other physical applications of the latter concern the scattering of an E-polarized electromagnetic wave by a conducting half plane as well as its acoustic counterpart.

0. Introduction

It is well known that the elastic SH (horizontally polarized shear) wave diffraction by a semi-infinite rigid ribbon [1,2], the diffraction of plane E-polarized (TE) electromagnetic waves by perfectly electrically conducting screen [3–5], and that of acoustic waves traveling in a fluid by sound-absorbing 'soft' screen [6], are mathematically identical. These problems belong to the category of two dimensional Helmholtz equation with Dirichlet boundary condition on a half line [7,8]. A large number of books and publications have attended to various aspects of this problem and its variants, over a long period of time [9–12,1,2]. Indeed, it was a century ago that A. Sommerfeld [13] solved the twin problems in diffraction theory for the two dimensional Helmholtz equation. One of these two problems deals with Dirichlet boundary condition ('soft surface') and the other deals with Neumann boundary condition ('hard surface') [7,12,1]. For complicated geometrical shapes and

E-mail address: bls@iitk.ac.in.

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boundary conditions, the two dimensional diffraction problem is posed on a suitably discretized domain and the traditional Helmholtz equation is replaced by an appropriate discrete counterpart [14–18].

This paper presents an analysis of a discrete analogue of the Sommerfeld 'soft' plane problem using the discrete Wiener–Hopf method [19–22], following Jones' approach [4,7]. The results and ensuing discussions form an inseparable part of the author's recent work on discrete Sommerfeld problems [23–25]. The discrete version of Sommerfeld 'soft' half plane problem is interpreted and formulated using a discrete Helmholtz equation on square lattice, based on 5-point discretization, with a semi-infinite rigid constraint. In this context, a study of discrete analogue of diffraction due to a semi-infinite 'hard' plane has been discussed using the same lattice model in [23]. In the language of mechanics, [23] provides an exact solution for diffraction on square lattice by a semi-infinite crack, using the discrete Wiener–Hopf method [19–21] and the discrete Fourier transform [26–28]. The overall approach is a fusion of tools developed by D. S. Jones [4,7] and L. I. Slepyan [29]. Indeed, the mechanical model of square lattice associated with the discrete formulation has been extensively studied earlier, for example, see the distinguished works [30,31] and [32,33,29] in the context of mechanics of a screw dislocation and Mode III crack, respectively. Several definitions and notational devices in [23], as well as the present paper, for the analysis on square lattice model, prior to and post the application of Fourier transform, are based on the applications expounded in [29]. In particular, the symbols $\mathscr{h}$, $\mathscr{r}$, $\mathscr{L}$, etc., have been used that occur frequently in the works of L. I. Slepyan [33,29]. The definition, and associated notation, of the discrete Fourier transform follows the classical works [26–28], as well as [29]. An interesting description of discrete Fourier transform is provided by Eatwell and Willis [34] 'as nothing more than a Fourier series *in reverse*.'

Although the square lattice formulation and discrete Fourier transform based Jones' method, as detailed in [23], has been adopted in this paper, still there are some outstanding issues which require an independent exposition, justifying the motivation behind this paper. For instance, the problem of crack on square lattice with nearest neighbor interaction involves only the crack opening displacement as unknown (as an obvious discrete analogue of the continuous case) while that discussed in this paper involves the displacement of particles *adjacent* to the constraint. Using the known scattered displacement at the constraint, the extension in the bonds adjacent to the constraint can be interpreted as an equivalent unknown entity, which then can be associated with a discrete analogue of the shear strain in continuous case [2,35] via difference approximation of a partial derivative. The significance of this innocuous observation is vital in a proper formulation allowing rigorous continuum limit (similar to that present in [24]). Due to the nature of nearest neighbor interactions in square lattice model, there is also an occurrence of unknown displacement of a particle ahead of the rigid constraint 'tip', since it is also adjacent to the constraint. The details of exact solution for the same, in fact, in closed form, are also provided in the paper. On the other hand, the governing equation is different for the particles at the crack face, while the problem of rigid constraint does not involve such issue. Note that both original Sommerfeld problems, with Dirichlet and Neumann boundary conditions on the half line, are also similar from the viewpoint of approach [7] but different in terms of details such as nature of the singularity of the convolution kernel, type of Sobolev space involved, etc., from the viewpoint of well-posedness [36]. The impact of the latter can be adjudged in the context of a well-posed continuum limit.

As an extension of the analysis presented in this paper, the operator-theoretic approach to diffraction on square lattice by a finite, as well as semi-infinite, rigid constraint is treated in [25]. This can be viewed as a discrete analogue of integral equation based framework that is well known for continuous case [37,38]. The questions, associated with the existence and uniqueness of solution in appropriate space, in connection with edge diffraction near any tip of a finite rigid constraint are addressed in [25]. A brief summary of the problem formulation, along with a list of associated definitions and several expressions, as well as the exact solution presented in Section 2 of this paper appear in Section 3 of [25] for quick references that is required for immediate application to near field analysis and operator-theoretic results therein. But the justification and details of manipulations behind that summary, for instance, arguments leading to the peculiar form of the Wiener–Hopf equation, its complete solution, etc., are provided in the present paper only. Also the far-field analysis and associated graphical results are not discussed in [25] but here, indicated by the titles of these papers.

As an added bonus of a detailed analysis of the discrete Sommerfeld problems on square lattice presented in this paper and [23], it has been found that there is a recurrence of several manipulations and techniques in formulation of problems, of the same type, on triangular [39] and hexagonal lattices [40], partly due to an advantage of certain notational choices that are carried over from square lattice formulation to these other lattices. The exact solution of rigid constraint diffraction problem on square lattice, that is attended in this paper, and that presented for the same problem on triangular lattice [39] have a strikingly close relationship, not shared by the corresponding crack diffraction problems on these two lattices.

0.1. Outline

The outline of the diffraction problem, and its solution, is as follows. After the formulation of square lattice model with a rigid constraint and the wave dispersion relation, the discrete Helmholtz equation is stated. Using a well defined discrete Fourier transform, along the rows of unbounded intact lattice, a general solution is constructed. Using this and the discrete Helmholtz equation on semi-infinite lattice row complementing the constrained half, the discrete Wiener–Hopf equation is derived as an inhomogeneous equation. The Fourier transform of the displacement of lattice row, located next to the constrained row, is present as an unknown function. A multiplicative factorization of the Wiener–Hopf kernel and additive factorization of the non-homogeneous term lead to the solution via an application of Liouville's theorem. The displacement of particle at lattice site facing the constraint tip, that also appears as an unknown in the problem, is determined subsequently.