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Characteristic fast marching method on triangular grids for the generalized eikonal equation in moving media

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HIGHLIGHTS

- Characteristic fast marching method (CFMM) for the generalized eikonal equation in moving media is extended to structured triangular grids.
- Numerical results are compared with the ray theory results and found to have good agreement.
- Presented a new result where the geometry of a wavefront propagating in a medium having Taylor–Green vortices is demonstrated.

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ABSTRACT

The governing equation of the first arrival time of a monotonically propagating front (wavefront or shock front) in an inhomogeneous moving medium is an anisotropic eikonal equation, called the *generalized eikonal equation* in moving media. When the ambient medium is at rest, this equation reduces to the well-known (isotropic) eikonal equation in which the characteristic direction coincides with the normal direction of the propagating front. The fast marching method is an efficient method for computing the first arrival time of a propagating front as the approximate solution of the isotropic eikonal equation. The fast marching method inherits the property that the characteristic direction coincides with the normal direction at every point on the propagating wavefront and therefore is well suited for the eikonal equation. Due to anisotropic nature, this property does not hold in the case of front propagation in a moving medium. Thus, the fast marching method cannot be directly used for the generalized eikonal equation and needs some suitable modifications. We recently proposed a characteristic fast marching method on a rectangular grid for the generalized eikonal equation (Dahiya et al., 2013) and shown numerically that this method is stable, accurate, and easy to update to second order approximations. In the present work, we generalize the method on structured triangular grids. We compare the numerical solution obtained using our method with the ray theory solution to show that the method captures accurately the viscosity solution of the generalized eikonal equation. We use the method to study some interesting geometrical features of an initially planar wavefront propagating in a medium with Taylor–Green type vortices.

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1. Introduction

The *generalized eikonal equation* (also called *anisotropic eikonal equation*) governing the first arrival time $T = T(\boldsymbol{x})$ of a front (wavefront or shock front) in a moving medium is given by (see Pierce [1])

$$\|\nabla T\|^2 - \frac{(1 - \boldsymbol{v}(\boldsymbol{x}) \cdot \nabla T(\boldsymbol{x}))^2}{(F(\boldsymbol{x}))^2} = 0, \quad \boldsymbol{x} = (x, y) \in \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is an open and connected set, $F = F(\boldsymbol{x})$ is the ambient sound speed and $\boldsymbol{v} = (v_1(\boldsymbol{x}), v_2(\boldsymbol{x}))$ is the ambient flow velocity. Throughout this work, we assume that $F(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \Omega$. Note that when the external velocity $\boldsymbol{v} = 0$, the generalized eikonal equation reduces to the (isotropic) eikonal equation

$$\|\nabla T\|^2 = \frac{1}{F^2}. \tag{2}$$

One of the early work of the generalized eikonal equation (1) is by Blokhintsev [2]. The generalized eikonal equation (1) is also obtained by Heller [3] in the context of the propagation of surface of acoustic discontinuities in moving media. For the derivation of concise wave equation in moving media and the generalized eikonal equation, we refer to Pierce [4]. Pierce [1] established (1) from a kinematical approach, which shows that this equation can be used to study the first arrival time of a front (either a wavefront or a shock front) in an arbitrary moving medium.

Generalized eikonal equation has many applications in areas like geosciences [5], study of atmospheric noise propagation [6–8], global monitoring of infrasound from natural and artificial sources [9]. The method based on tracing rays from each point on the wavefront (called, the ray tracing method in geometrical optics theory) is a classical way of obtaining the solution to (1). This is basically a characteristic method which, in general, cannot be used to obtain the solution globally as the solution may become multivalued at some points sufficiently away from the initial front. This is due to the fact that for certain boundary conditions, the rays converge and form a caustic region beyond which the propagating wavefront folds. In the caustic region, we need to interpret the solution in a weak sense called the viscosity solution as suggested by Crandall and Lions [10].

In general, we cannot obtain an explicit solution to the eikonal equation, especially in anisotropic cases. Qian et al. [11] developed a level set based algorithm for anisotropic wave propagation in the caustic region. Some well-known numerical methods for computing the viscosity solution of anisotropic eikonal equations are Fast sweeping methods [12–14], the ordered upwind method [15], the buffered fast marching method [16], and the expanding wavefront method [17]. These methods are also extended to general static Hamilton–Jacobi type equations of the form

$$H(\nabla T, \boldsymbol{x}) = 0 \tag{3}$$

in anisotropic media. The generalized eikonal equation (1) is a particular case of (3) and therefore these methods may be used to approximate the viscosity solution of this equation. Gremaud and Kuster [18] (also see Alton and Mitchell [19]) compared the fast marching method of Sethian [20] and the fast sweeping method (see for instance Zhao [21]), and concluded that the fast sweeping method is less efficient than the fast marching method for (3) when the characteristic curve of this equation is more curved. This is typically the case for (1) when the magnitude of the external velocity $\boldsymbol{v}$ is sufficiently large. Hence, in such cases, fast sweeping methods are less efficient for the generalized eikonal equation (1). The ordered upwind method, the buffered fast marching method, and the expanding wavefront method generally involve a nonlinear equation to be solved in order to obtain the solution of (3). In the case of (1), these nonlinear equations need an additional nonlinear solver like Newton–Raphson's method which will increase the computational cost of the method.

Recently, Dahiya et al. [22] developed the characteristic fast marching method (CFMM) (in rectangular grids) for the generalized eikonal equation (1), where they showed that the method is robust in computing the viscosity solution even when the external velocity is not axially aligned. The method is very simple as it involves a quadratic equation which can be solved explicitly without using any nonlinear solver. In CFMM, we also use the trial values (values in the narrow band points) in the computation and use the quadrant indicator map to keep track of those points which can be acceptable. Whereas, the ordered upwind method of Sethian and Vladimirsky [15] never uses the trial values, but uses only the alive points through the concept of alive fronts and narrow band fronts. This makes our method simpler to implement than the ordered upwind method. The computational complexity of our algorithm is the same as that of the fast marching method and is given by $O(N \log N)$, where N is the total number of grid points. For the implementation of the ordered upwind method to the generalized eikonal equation in the context of modeling folding in rocks, we refer to Hjelle et al. [23], where the authors also obtained quadratic equation as part of their scheme.

The CFMM is developed and tested by Dahiya et al. [22] on the rectangular grids. But in many practical situations (see for instance, Rawlinson and Sambridge [24]), the medium under consideration may involve either complicated geometry and/or the speed function F (and/or the external velocity $\boldsymbol{v}$) may be discontinuous with a nonlinear curve of discontinuity. In such cases, using triangular grids is more convenient than the rectangular grids, especially near the boundary of the medium and/or the curve of discontinuity of F (and/or $\boldsymbol{v}$). The fast marching method for the eikonal equation (2) was first extended to triangular grids by Kimmel and Sethian [25] (also see Barth and Sethian [26]), and the triangulated domain version of the fast sweeping method for (2) was first given by Qian et al. [27]. Other fast methods like generalized fast marching method (see