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Nonlinear wave propagation and reflection—Comparing the numerics with the analytics

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HIGHLIGHTS

- The nonlinear wave equation is solved analytically by the perturbation method.
- The analytical results are compared with the numerical solution.
- Exact description of the interaction region for counter-propagating waves is given.

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ABSTRACT

The accuracy of numerical methods needs always a special attention. In this paper, analytical and numerical methods have been compared to describe the initial stage of nonlinear propagation and reflection of longitudinal ultrasonic waves. The perturbation method has been used to derive the analytical solution and the finite difference scheme to find the numerical solution for multiple free-boundary reflections of a harmonic burst at ultrasonic frequencies. The comparison of results at relatively small nonlinearities reveals a good qualitative and quantitative agreement between the analytical and numerical solutions. The method for determining analytically the exact region of interaction for counter-propagating waves is outlined in detail. At higher frequencies and larger nonlinear effects some quantitative differences between analytical and numerical results appear. The results are applicable in modelling nonlinear wave motion, including NDT and nonlinear one-dimensional vibrations.

1. Introduction

The studies of one-dimensional nonlinear longitudinal wave propagation and reflection at ultrasonic frequencies are not new, for earlier work papers by Breazeale [1], Thompson [2] or books by Engelbrecht [3], Nigul [4] and Davison [5] may be cited among many. In more recent studies the nonlinear effects have remain topical in relation to wave motion in inhomogeneous materials [6] or in the studies of time-reversed acoustics [7].

As an example, the theoretical results presented in this paper may be compared with a paper by Thompson et al. [2]—a short paper on experimental results that describe the effect of ultrasonic nonlinear wave propagation in solids with multiple free-boundary reflections. The characteristic of this phenomenon is that at the initial stages of propagation of a finite-amplitude harmonic wave there is an emergence of the second harmonic which amplitude increases linearly by the distance travelled. But after reflection from the free-boundary the back-propagating wave has a linearly decreasing amplitude so that when the wave arrives at the boundary where it was generated the amplitude for the second harmonic is zero. After

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1 Research on this topic began as a joint work with a dear friend and colleague Arvi Ravasoo (1939–2014).

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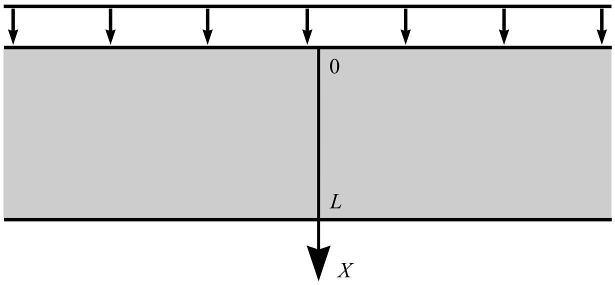

Fig. 1. Specimen with two parallel surfaces.

the second reflection and forward-propagation the amplitude starts to increase linearly again [2]. These effects have been topical much more recently in relation to nonlinear time-reversed acoustics [7] when observing time-reversal invariance of a nonlinear harmonic wave before shock-wave formation.

In the current paper the nonlinear wave propagation with multiple free-boundary reflections has been analysed both analytically by the perturbation method and numerically by making use of the finite difference scheme. The comparison of the analytical with the numerical results reproduces the nonlinear effects cited above. In addition, it is revealed that near the boundaries where propagating and reflecting waves interact occurs a change in the phase velocity. These changes can be seen in the comparison with the numerical results and an analytical formula has been found to address the changes in phase velocities during interaction. All numerical and analytical calculations presented are carried out by the aid of software package *Maple* 9.

The outlined results are important by pointing out the limits and possibilities of the perturbation analysis in describing nonlinear wave phenomenon. Also, in a special case the proposed analytical solution for a harmonic burst generates a state of nonlinear one-dimensional vibration that is applicable in NDT.

2. Problem formulation

A homogeneous elastic layer is considered, see Fig. 1. A longitudinal wave motion of ultrasonic frequency is excited at the boundary $X = 0$ of the specimen and the process is followed until the wave front has made two round trips along the material thickness L including three free-boundary reflections. It is assumed to be an initial stage of nonlinear wave propagation where wave amplitude is small but finite and no shock waves are generated.

The wave motion is governed by one-dimensional nonlinear equation of motion [1,4,6,8]:

$$[\,1 + k\,U_X(X,t)\,]\,U_{XX}(X,t) - c^{-2}\,U_{tt}(X,t) = 0, \tag{1}$$

where $U(X,t)$ is the material particle displacement, X Lagrangian coordinate, t is time. Double or single variable notation in the indices XX, tt and X represents second or first order partial derivatives, respectively. In Eq. (1) c represents the linear phase velocity and k is the nonlinearity parameter. These quantities are related to the linear elastic constant α, the nonlinear elastic constant β and the density ρ_0 as:

$$c = \sqrt{\frac{\alpha}{\rho_0}}, \qquad k = 3\left[1 + \frac{\beta}{\alpha}\right]. \tag{2}$$

The linear and nonlinear elastic constants are related to the five-constant nonlinear theory of elasticity as:

$$\alpha = \lambda + 2\mu, \qquad \beta = 2\,[\nu_1 + \nu_2 + \nu_3]\,. \tag{3}$$

Here λ and μ are the usual Lamé or second order elastic constants and ν_1, ν_2, ν_3 are the third order elastic constants as used by Bland [9] and Nigul [4].

The initial and boundary conditions to solve Eq. (1) for multiple free-boundary reflections are:

$$\begin{aligned} &U(X,0) = U_t(X,0) = 0, \\ &U_X(0,t) = [H(t) - H(t-t_0)]\,\varepsilon\,\sin(\omega t), \qquad U_X(L,t) = 0, \end{aligned} \tag{4}$$

i.e. particle displacements and velocities are initially zero and the boundary conditions in (4) describe the free-boundary condition at $X = L$ and a harmonic burst type excitation at the boundary $X = 0$ given for $U_X(X,t)$. $H(t)$ and $H(t-t_0)$ represent the Heaviside step functions, ω is the radial frequency, $\varepsilon \ll 1$ represents the initial wave amplitude and t_0 is a finite time length chosen so that the harmonic burst includes exactly integer number of periods of oscillation, i.e. $t_0 = 2\pi n/\omega$ $(n = 1, 2, 3, \ldots)$. In addition, the time length t_0 is no larger than the time it takes for the phase front to travel with linear phase velocity c one round trip back and forth within the specimen thickness, i.e. $t_0 \leq 2L/c$. For the reflected wave returning to the boundary $X = 0$ this condition guarantees that the reflection at $X = 0$ is a free-boundary reflection.

If wave propagation in one direction is considered then up to the point of shock-wave formation the equation of motion Eq. (1) does have a known analytical solution in the form of special functions [1], but in the case of propagation in a finite interval with boundary reflections the authors are not aware of any general analytical solutions.