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On functional equations leading to exact solutions for standing internal waves

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HIGHLIGHTS

- Standing internal waves below a horizontal plane are described by Schröder functional equations.
- We give a unified approach to many exact solutions for standing internal waves below a horizontal plane.
- Relevant results on Schröder and Abel functional equations are presented and used.

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ABSTRACT

The Dirichlet problem for the wave equation is a classical example of a problem which is ill-posed. Nevertheless, it has been used to model internal waves oscillating harmonically in time, in various situations, standing internal waves amongst them. We consider internal waves in two-dimensional domains bounded above by the plane $z = 0$ and below by $z = -d(x)$ for depth functions d. This paper draws attention to the Abel and Schröder functional equations which arise in this problem and use them as a convenient way of organising analytical solutions. Exact internal wave solutions are constructed for a selected number of simple depth functions d.

1. Introduction

Internal gravity waves form the final chapter of a classic book on “Waves in Fluids” [1]. Equation (22) at [1] states that the upward component of the mass flux, denoted there by q but here by w, satisfies

$$\Delta\left(\frac{\partial^2 w}{\partial t^2}\right) = -N(z)^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right),$$

where Δ is the 3-dimensional Laplacian, and z is the vertical coordinate. Here $N(z)$ is the Brunt–Väisälä frequency. For 2-dimensional flows, i.e. no y dependence, there is a stream function, and several problems of physical interest involve solutions of the form $w(x, z, t) = \psi(x, z)\exp(i\omega t)$, and when, additionally, the Brunt–Väisälä frequency is constant, ψ satisfies the one-dimensional wave equation in the space variables. (See Eq. (2.1).)

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The problem we treat in this paper – standing internal waves – is ill-posed, and, in particular, solutions when they exist are not unique. The same pde but with different boundary conditions describes two-dimensional internal waves generated by an oscillating cylinder in a uniformly stratified fluid and a few comments on such local wave generation are given in our Section 9. A photograph of the wave pattern for local wave generation is given in Figure 76 on page 314 of [1] and a diagram indicating the beams of internal waves is given in Figure 2 of [2]. The characteristic directions of the pde are very evident. For our standing wave problem, once again the characteristic directions are often evident in the flow fields: see, for example, our Fig. 3 and other publications on the subject, including photographs of experiments.

For general plane domains standing waves are treated in [3]: see the sections in [3] starting with that on Sobolev's equation. In this paper we specialise to fluid domains confined by a flat surface $z = 0$ and a bottom boundary $z = -d(x)$ for a given non-negative depth function d. Exact solutions for certain depth functions d are known, e.g. Wunsch's solution for a subcritical wedge [4], Barcilon's solution in a semi-ellipse [5] and a self-similar solution in a specific trapezoid [6], among many others. It is known that analytical solutions to the wave equation (2.1) with Dirichlet boundary conditions can be constructed from functions which satisfy the functional equation

$$f\left(x+\frac{d(x)}{\nu}\right)=f\left(x-\frac{d(x)}{\nu}\right)+Q,$$

for $\nu > 0$ and Q given constants. Of course, when $Q > 0$ the preceding equation can be scaled and if a solves Eq. (1.1a), then Qa will solve the preceding equation. Results concerning the following linear functional equations are central to our study of standing internal waves:

$$a\left(x+\frac{d(x)}{\nu}\right)=a\left(x-\frac{d(x)}{\nu}\right)+1, \tag{1.1a}$$

$$f\left(x+\frac{d(x)}{\nu}\right)=f\left(x-\frac{d(x)}{\nu}\right). \tag{1.1b}$$

These functional equations have been used for internal wave studies for several decades: see [7] and references therein. The physical interpretation of Q non-zero is a constant mass-flux through the domain and it is considered in [8,9] in the context of tidal conversion. The zero-flux boundary condition $Q = 0$ as in Eq. (1.1b) is the physical condition appropriate to standing waves (and blinking modes) and is the main topic of this article. It has been noticed by [10] (their Theorem 2) and [11] that there are reformulations of Eq. (1.1b) such that one can associate solutions to Eq. (1.1b) with solutions to Eq. (1.1a). However, to date, very little use of advantages associated with these reformulations seems to have been made in the construction of analytical internal wave solutions.

For a large class of depth functions d one can invert the arguments in the functional equations (1.1) and formulate them as the functional equations (3.1) presented in Section 3, which corresponds to a special case of Schröder's functional equation for $Q = 0$ and Abel's functional equation $Q \neq 0$. The Schröder and Abel functional equations are well-studied functional equations [12,13]. In this article known properties of these functional equations are put into context for the construction of internal waves. A selection of analytical internal wave solutions constructed from solutions to these functional equations is presented. Besides the application to internal waves, there are other wave phenomena described by the same boundary-value problem: we mention some of these at the end of Section 2.

The structure of this paper is as follows. In Section 2 we present the partial differential equation boundary-value problem that models the internal waves and in Section 3 we present the corresponding functional equations. We present in Section 4 Wunsch's solution for a subcritical wedge, and follow this in Section 5 with various solutions for standing waves with everywhere subcritical bottom profiles. Our treatment in Section 6 and in Section 7 indicates results for bottom profiles that have some supercritical parts. The latter of these two sections, Section 7, treats a particularly simple solution method appropriate when d is related in a certain way to involutions. We are confident that the methods allow both for further application and further development. The question of what other wave problems lead to similar functional equations, a topic which takes us away from internal waves, is addressed in Section 8. We return to internal waves in Section 9 and propose related problems where the functional equation methods might be used.

There is no claim that any new solutions in this paper – or indeed any other solutions from our functional equation approach – can only be obtained by the methods of this paper. Our paper is an exposition of the easier results associated with the functional equations (1.1) and (3.1), and we hope that others will develop the approach. We expect that future developments are most likely to be useful in establishing general qualitative aspects of the solutions. For the present, we wish to remind researchers in the area of the spectacular nonuniqueness of solutions, and the methods of generating more, as given in Theorem 2. This result and some others in this paper are given in [11], albeit without noting the relation to the standard functional equation literature. We expect future developments will treat 'attractor' solutions, as in [7,6] and will establish results, particularising to domains with $z = 0$ as part of their boundary, using functional-equation and dynamical-systems approaches as in [3]. These matters concern bottom profiles which contain both subcritical and supercritical parts (as defined in Section 2.1) and situations where for some values of ν the only solution is the zero flow solution (f is constant); then, as exemplified in Section 6.1 one is required to determine for which values of ν there are nontrivial solutions, and find f then. We have chosen to organise our paper around a selection of exact solutions as, despite the large