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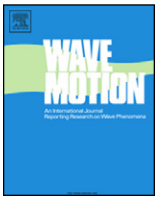

Study of anomalous wave propagation and reflection in semi-infinite elastic metamaterials

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HIGHLIGHTS

- We study wave propagation in semi-infinite elastic metamaterials.
- A complete wave mode conversion is numerically demonstrated.
- Negative refraction of elastic waves is captured by the generalized Snell's law.
- Effects of negative properties on wave propagation are illustrated.

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ABSTRACT

Elastic metamaterials have been investigated to achieve negative effective properties, which cannot be found in the conventional elastic medium. In this paper, plane wave propagation and reflection in semi-infinite elastic metamaterials with doubly or triply negative material properties are studied analytically and numerically. The unique negative refractions for the longitudinal (P) wave and transverse (S) wave are captured by the proposed generalized Snell's law. Attention is paid to quantitative characterization of the effects of different negative property combinations on the anomalous wave propagation. The effects of different angles of incidence are also investigated for both double-negative and triple-negative transmitted media and some unusual wave propagation phenomena such as complete wave mode conversion are numerically demonstrated. This study can serve as the theoretical foundation for engineering and designing general metamaterial-based elastic wave devices.

1. Introduction

Metamaterials have become a subject of great interest because of their abilities to achieve negative effective material properties unlike those of any conventional materials by introducing subwavelength resonators into their building blocks [1–7]. Double-negative electromagnetic (EM) metamaterials denote those artificial structures in which both the dielectric constant and magnetic permeability are simultaneously negative within a certain frequency regime [1,2]. Recently, the acoustic metamaterial, an analogue of the EM metamaterial, has also received considerable attention due to its exotic

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acoustic properties such as negative effective mass density and/or bulk modulus. These metamaterials are compelling in view of the possibility of using negative refraction to design flat superlenses that can beat the diffraction limit [3].

Different from the acoustic metamaterial, the elastic metamaterial is a structured composite with a solid host matrix, which can support both longitudinal and transverse waves. Because of the vector characteristics of elastic waves and the possible coupling between the longitudinal and transverse wave modes, richer wave propagation phenomena are expected in the elastic metamaterial. Three independent effective parameters are required to completely characterize the solid material: effective bulk modulus, effective shear modulus, and effective mass density. The negative values of each single effective parameter and their various combinations can lead to eight kinds of elastic materials, which is twice as many as those in the EM and acoustic cases. Different microstructures were proposed to achieve the negative effective mass density and modulus, respectively. The negative effective mass density was experimentally demonstrated through a locally resonant structure by embedding soft silicon rubber coated lead spheres into an epoxy matrix medium [4]. The negative modulus, on the other hand, was demonstrated through an array of subwavelength Helmholtz resonators [5]. Extended theoretical studies on negative effective material properties were conducted by Milton and Willis [6] through a mass–spring model and by Movchan and Slepyan [7]. Dipolar and monopolar local resonances are regarded as the interior wave mechanisms for the negative mass density and the negative bulk modulus of the elastic metamaterial, respectively [8,9]. Liu et al. [10] first proposed a chiral elastic metamaterial with simultaneously translational and rotational resonances to achieve negative mass density and negative bulk modulus in certain frequency ranges. Negative refraction of a double-negative elastic metamaterial with single-phase chiral microstructure was experimentally demonstrated [11]. A metamaterial comprised of a fluid–solid multi-phase composite to enhance the quadrupolar resonance was suggested to possess simultaneously negative mass density and negative shear modulus and therefore, achieve negative refraction over a large frequency region [12]. Discrete structural interfaces with negative properties were also proposed to achieve negative refraction and flat lens focusing of elastic waves [13]. Most recently, Antonakakis et al. [14] presented high frequency asymptotic analysis for elastic waves leading to dynamic effective properties. Various novel concepts and engineering applications of elastic metamaterials have been successfully demonstrated such as seismic wave filters, sound and vibration isolators, elastic waveguides and energy harvesters [15–21].

Elastic wave propagation in solid media is an old subject and fruitful results and techniques can be found in many classic wave propagation books [22,23]. However, the introduction of elastic metamaterials broadens the horizon of this subject. In order to study the anomalous wave behavior and explore related engineering applications of elastic metamaterials, there is a great desire to investigate the wave propagation and reflection behavior in semi-infinite elastic media with various negative material properties. In this paper, plane wave propagation and reflection involving semi-infinite elastic media with doubly or triply negative properties are studied analytically and numerically. Various combinations of positive and negative material properties in transmitted elastic media are considered for different angles of incidence. Anomalous wave phenomena, such as negative refractions and complete wave mode conversion between the longitudinal and transverse waves, are investigated. The study can serve as the theoretical foundation for engineering and designing general metamaterial-based elastic wave devices. The paper is arranged in four sections including this introduction: in Section 2, the plane wave propagation involving the semi-infinite elastic metamaterial with negative material properties and standard positive elastic medium is analytically formulated; in Section 3, numerical results and related discussions for the anomalous wave reflection and refraction are presented. The effects of different negative property combinations on the anomalous wave propagation is quantitatively investigated for various angles of incidence. Finally, conclusions are presented in Section 4.

2. Plane wave propagation and reflection in the elastic media with negative material properties

A linear isotropic elastic medium can be characterized by Lame constants λ, μ and the mass density ρ. In two dimensional (2D) plane strain problems, the P wave modulus is $E_M = \lambda + 2\mu$ [24] and μ is the shear wave modulus. Therefore, wave dispersion relations of the P and S waves can be determined by:

$$\boldsymbol{k} \cdot \boldsymbol{k} = \begin{cases} \dfrac{\omega^2 \rho}{E_M}, & \text{for the P wave} \\ \dfrac{\omega^2 \rho}{\mu}, & \text{for the S wave} \end{cases} \tag{1}$$

where $\boldsymbol{k}$ is the wave vector and ω is the angular frequency. Based on the definition, the magnitude of wave vector $\boldsymbol{k}$ should be an imaginary number if the elastic medium possesses positive modulus and negative mass density, which implies that the elastic wave is prohibited in such medium. However, the magnitude of wave vector $\boldsymbol{k}$ can be a positive real number when both the modulus and density are negative, which means that the elastic waves can propagate in the medium with double-negative or triple-negative material properties. The energy flow in the elastic medium can be described by the Poynting vector $\boldsymbol{P}$ as:

$$\boldsymbol{P} = -\boldsymbol{v}^* \cdot \boldsymbol{\sigma}, \tag{2}$$

where $\boldsymbol{\sigma}$ is the stress field tensor, $\boldsymbol{v}$ is the particle velocity vector and the asterisk denotes a complex conjugate. Without loss of generality, let us consider only the P wave propagation [25]:

$$\boldsymbol{k} \cdot \boldsymbol{P} = \frac{1}{2} \omega \rho \left| \boldsymbol{v_p} \right|^2 \tag{3}$$