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Slope modulation of ring waves governed by two-dimensional sine–Gordon equation

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HIGHLIGHTS

- A theory of slope modulation of waves governed by the 2-D sine–Gordon equation is proposed.
- A large time asymptotic solution describing the slope modulation of trains of axi-symmetric kinks is obtained.
- The comparison with the numerical solution of 2-D sine–Gordon equation shows excellent agreement.

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ABSTRACT

A theory of slope modulation of waves governed by the two-dimensional sine–Gordon equation is proposed. A large time asymptotic solution describing the slope modulation of trains of ring kinks is obtained. The comparison with the numerical solution of two-dimensional sine–Gordon equation shows excellent agreement.

1. Introduction

Sine–Gordon equation, occurring in various problems of mathematical physics [1,2], has attracted attention of mathematicians and physicists since the seventies of the 20th century due to the possibility of exact integration by the inverse scattering transform [3,4], or, alternatively, by the direct method [5]. Around the same time, Whitham [6,7] initiated the variational method which enables one to study the asymptotic theory of amplitude modulation of a non-linear wave packet. Whitham's method has subsequently been developed by many researchers and has found various applications in propagation of nonlinear dispersive waves (see, e.g., [8–10] and the references therein). However, this method requires the knowledge about the uniform solution (or its first integral) of the corresponding differential equation in one wavelength, so it cannot be extended to systems where such uniform solutions are not known in advance. In contrary, the variational-asymptotic method, proposed first by Berdichevsky in [11] (see also [12,13]), develops the systematic multi-scale averaging procedure for variational problems of wave propagation containing small parameters which enables one to reduce them to the average variational problems via the solution of the so-called strip problems. The aim of this short note is to apply the variational-asymptotic method to derive the equation of slope modulation for wave packets of multi-kink governed by two-dimensional sine–Gordon equation and extend the results obtained in [14] to the two-dimensional case. In contrast to

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its counterpart in one dimension, the 2-D sine–Gordon equation is non-integrable in general [15,16]. Numerical simulations of the ring-shaped kink-solution (see [17–19] and the references therein) discovered interesting behaviors like a return effect and a reflection of waves at the origin (see also the similar studies for the ring-shaped soliton obeying the nonlinear Schrödinger equation, called ring dark soliton, in [20–22]). However, as the numerical simulation for multi-kink near the origin is quite difficult due to the singularity of the governing differential equation, it remains still unclear about the interaction and collective behavior of multi-kink at large time during the return period and after the reflection about the origin. We shall integrate the 2-D equation of slope modulation to find an exact analytical solution for the ring-shaped multi-kink that delivers useful information about the collective behavior of kinks.

2. Theory of slope modulation for 2-D sine–Gordon equation

The 2-D sine–Gordon equation

$$\varphi_{tt} - \Delta\varphi + \sin\varphi = 0 \quad (\Delta = \partial_1^2 + \partial_2^2)$$

can be obtained from the stationarity of the following two-dimensional functional

$$I[\varphi(\mathbf{x}, t)] = \iint \left[\frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}\varphi_t^2 - (\cos\varphi - 1)\right] d\mathbf{x}dt. \tag{1}$$

We look for the extremal of this variational problem in the form of a slowly varying wave packet

$$\varphi = \phi(\theta(\mathbf{x}, t), \mathbf{x}, t), \tag{2}$$

where ϕ is a function of three arguments θ, $\mathbf{x}$, and t satisfying the following condition

$$\phi(\theta + 2\pi, \mathbf{x}, t) = \phi(\theta, \mathbf{x}, t) + 2\pi.$$

The fast variable $\theta(\mathbf{x}, t)$ plays the role of the phase, with $\nabla\theta$ and $-\theta_t$ corresponding to the wave vector $\mathbf{k}$ and the frequency ω, respectively. The above condition means that the train of 2π-kinks is being dealt with. We assume that the partial derivatives $\partial_{\mathbf{x}}\phi$ and $\partial_t\phi$ at fixed θ are negligibly small compared with $\phi_\theta\nabla\theta$ and $\phi_\theta\theta_t$, respectively. Besides, the wave number and the frequency change slowly in one wave length and one period. The precise meaning of these assumptions can be given in terms of the characteristic length- and time-scales (cf. [13]). Thus, in the first approximation

$$\nabla\varphi = \phi_\theta\nabla\theta, \qquad \varphi_t = \phi_\theta\theta_t.$$

Substituting these into (1), we obtain the functional

$$I_0[\phi, \theta] = \iint \left\{\frac{1}{2}[(\nabla\theta)^2 - \theta_t^2]\phi_\theta^2 - (\cos\phi - 1)\right\} d\mathbf{x}\, dt.$$

Following [13], we formulate the strip problem as follows: Find the extremal of the functional

$$\frac{1}{2\pi}\int_0^{2\pi} \left[\frac{1}{2}m\phi_\theta^2 - (\cos\phi - 1)\right] d\theta, \tag{3}$$

with $m = \mathbf{k}^2 - \omega^2$, among functions $\phi(\theta)$ satisfying the conditions

$$\phi(2\pi) = \phi(0) + 2\pi, \qquad \phi_\theta(2\pi) = \phi_\theta(0), \qquad \max_{\theta\in[0,2\pi]} |\phi_\theta| = p. \tag{4}$$

The last condition means that the maximal slope ϕ_θ is fixed in this strip problem.

Denote by L the extremum of the functional (3), which is a function of p, $\mathbf{k} = \nabla\theta$ and $-\omega = \theta_t$, with L being called average Lagrangian. In the first approximation the variational problem (1) reduces to the stationarity of the average functional

$$\bar{I}_0[p, \theta] = \iint L(p, \nabla\theta, \theta_t) d\mathbf{x}dt.$$

Euler–Lagrange's equations of this functional read

$$\frac{\partial L}{\partial p} = 0, \qquad \frac{\partial}{\partial t}\frac{\partial L}{\partial \theta_t} + \nabla\cdot\frac{\partial L}{\partial\nabla\theta} = 0. \tag{5}$$

Eq. $(5)_1$ expresses the solvability condition for the strip problem in the form of the nonlinear dispersion relation, while $(5)_2$ is the equation of slope modulation.

Using the obvious first integral in the strip problem

$$\frac{1}{2}m\phi_\theta^2 + (\cos\phi - 1) = h,$$

we find the solution in terms of elliptic functions and then the average Lagrangian according to

$$L = \frac{1}{2\pi}\int_0^{2\pi} m\phi_\theta^2 d\theta - h = \frac{1}{2\pi}\int_0^{2\pi} m\phi_\theta d\phi - h.$$