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# Improved multimodal methods for the acoustic propagation in waveguides with finite wall impedance

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## HIGHLIGHTS

- We address the problem of acoustic propagation in waveguides with treated boundaries.
- Two improved multimodal formulations are proposed and compared.
- Both formulations significantly increase the convergence of the modal method.
- The formulation with a supplementary mode is found to be the most efficient.

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## ABSTRACT

We address the problem of acoustic propagation in waveguides with wall impedance, or Robin, boundary condition. Two improved multimodal methods are developed to remedy the problem of the low convergence of the series in the standard modal approach. In the first improved method, the series is enriched with an additional mode, which is thought to be able to restore the right boundary condition. The second improved method consists in a reformulation of the expansions able to restore the right boundary conditions for any truncation, similar to polynomial subtraction technique. Surprisingly, the first improved method is found to be the most efficient. Notably, the convergence of the scattering properties is increased from $N^{-1}$ in the standard modal method to $N^{-3}$ in the reformulation and $N^{-5}$ in the formulation with a supplementary mode. The improved methods are shown to be of particular interest when surface waves are generated near the impedance wall.


## 1. Introduction

We address the problem of acoustic wave propagation problem, $(\Delta + k^2)p(x, y) = 0$, within a waveguide with, locally, an impedance boundary condition at the wall $(y = h)$:

$$\partial_y p(x, h) = \frac{1}{Z(x)} p(x, h), \quad x \in [0, L] \tag{1}$$

with $Z(x)$ the surface impedance (Fig. 1). This impedance condition, also referred as Robin condition [1–5], is of practical interest, since it describes non perfectly reflecting surfaces or absorbing materials at a waveguide wall. Limiting cases are the Neumann boundary condition $Z = \infty$ and the Dirichlet boundary condition $Z = 0$.

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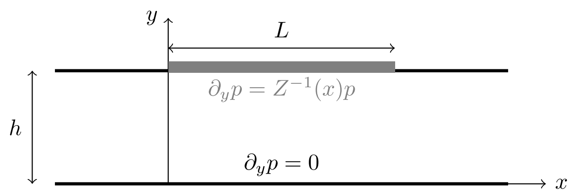


**Fig. 1.** Waveguide with the upper wall submitted to a localized Robin boundary condition ($0 \le x \le L$), Neumann boundary condition otherwise.

**Table 1**
Convergences of the remaining series, of the scattering coefficients and of the scattered field in the standard multimodal formulation (SMF), in the reformulation (RF) and in the formulation with a supplementary mode (SupMF).

| | Standard modal formulation | Reformulation | Supplementary mode |
|---|---|---|---|
| Remaining series | $1/N^{1.5}$ | $1/N^{3.5}$ | $1/N^{3.5}$ |
| Error on the scattering coef. | $1/N$ | $1/N^3$ | $1/N^5$ |
| Error on the scattered field | $1/N$ | $1/N^3$ | $1/N^{3.5}$ |

Classically, that is, in waveguides with rigid boundaries, multimodal formulations involve the expansion of the solution on the *rigid* transverse modes $\varphi_n(y)$ satisfying the Neumann boundary condition:

$$p(x, y) = \sum_{n=0}^{N} p_n(x)\varphi_n(y). \tag{2}$$

Since the infinite set of $\varphi_n, N \to \infty$, is a complete basis, the decomposition is still valid in segments with Robin boundary condition at the walls, as done in [6,7]. However, because the boundary condition is not satisfied by the transverse modes, the series has poor convergence, attributable in part to the non uniform convergence of the pressure derivative. This results in the appearance of Gibbs oscillations close to the treated wall. In the case of waveguides with varying cross-section, it has been shown in Refs. [8–11] that this situation can be remedied by adding to the usual expansion an additional term (called supplementary mode in the sequel):

$$p(x, y) = \sum_{n=0}^{N-1} p_n(x)\varphi_n(y) + p_{-1}(x)\varphi_{-1}(y), \tag{3}$$

where $\varphi_{-1}$ is chosen such that $\varphi'_{-1}(h) \neq 0$ (note that this ensures that $\varphi_{-1}$ is not a finite combination of the $\{\varphi_n\}_{n\geq 0}$, otherwise $\varphi_{-1}$ would be trivially absorbed into the $\varphi_n$-series above a given order). In this approach, it is thought that the supplementary mode will be able to restore the right boundary condition if $Z(x)p_{-1}(x)\varphi'_{-1}(h) = p(x, h)$. However, this is not guaranteed *a priori*; in the case of waveguides with varying cross section, it has been shown that the right boundary condition is restored only asymptotically, for $N \to \infty$ [9].

Alternatively to this supplementary mode, Bi et al. [12] proposed a reformulation of the modal expansion that restores the exact boundary condition for any truncation of the series. Instead of using an additional unknown $p_{-1}(x)$, the projection is written

$$p(x, y) = \sum_{n=0}^{N} p_n(x)[\varphi_n(y) + Y(x)\varphi_n(h)\xi(y)] \tag{4}$$

with $Y(x) = Z^{-1}(x)$ the surface admittance. With $\xi(y)$ being chosen such as $\xi(h) = 0$ and $\xi'(h) = 1$, it is easy to see that the condition $\partial_y p(x, h) = Y(x)p(x, h)$ is satisfied for any $N$ value. Finally, the function $\xi(y)$ is chosen in order to ensure that the truncated series satisfies the Neumann boundary condition at $y = 0$, thus such that $\xi'(0) = 0$. This reformulation is similar to the so-called polynomial subtraction method [13–15] (see also [9,10] for a discussion in the 2D-case).

Note that, in Eqs. (2) and (3), the $p_n$-functions correspond to the usual modal components $p_n = (p, \varphi_n)$, with $(f, g)$ the scalar product $\int_0^h \mathrm{d}y\ f(y)\bar{g}(y)$. However, in Eq. (4), they are defined as the modal component of a related field $\tilde{p}$ ($p_n \equiv (\tilde{p}, \varphi_n)$), with $\tilde{p}(x, y) \equiv p(x, y) - Y(x)p(x, h)\xi(y)$ (see Section 2.2).

In this paper, we compare the two improved multimodal approaches. As expected, both formulations lead to an increased convergence. However, one may think that the expansion proposed in the reformulation, Eq. (4), that exactly satisfies the boundary condition (1), is the most efficient because it ensures the uniform convergence of the derivative of the series. In fact, both formulations give similar convergence rate and accuracy for the wavefield in the scattering region. More surprisingly, when computing the scattering coefficients, the "supplementary mode" formulation (SupMF) is characterized by a superconvergence: while the standard modal expansion converges as $1/N$ and the reformulation (4) as $1/N^3$, it indeed displays a $1/N^5$ convergence rate. Summarized convergence properties are presented in Table 1.

The paper is organized as follows. In Section 2, the multimodal formulations issued from the expansions in Eqs. (2)–(4) are derived. Results on the convergence are presented in Section 3. Technical calculations are collected in the appendices.