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# Hamiltonian formulation of 2 bounded immiscible media with constant non-zero vorticities and a common interface

Alan Compelli

*School of Mathematical Sciences, Dublin Institute of Technology, Kevin Street, Dublin 8, Ireland*

## HIGHLIGHTS

- A Bernoulli condition and kinematic boundary conditions are derived for a bound 2 media rotational water-wave system with free common interface
- Non-canonical Hamiltonian equations of motion are derived
- Under a variable transformation the equations are presented in a canonical Hamiltonian form

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## ABSTRACT

We examine a 2-dimensional water-wave system, with gravitationally induced waves, consisting of a lower medium bound underneath by an impermeable flat bed and an upper medium bound above by an impermeable lid such that the 2 media have a free common interface. Both media have constant density and constant (non-zero) vorticity. By examining the governing equations of the system we calculate the Hamiltonian of the system in terms of its conjugate variables and perform a variable transformation to show that it has canonical Hamiltonian structure.



## 1. Introduction

In 1968 Zakharov published a paper [1] showing the canonical Hamiltonian structure of an infinitely deep irrotational fluid system, i.e. with zero vorticity, with a free surface with gravitationally induced waves. Further relevant studies of the irrotational case were carried out in [2–6]. At the beginning of the 19th century Gerstner [7] had studied vorticity and more recently there have been several papers of interest which consider the rotational case, i.e. with non-zero constant vorticity, e.g. [8–17]. In particular Constantin et al. [18] showed that a consideration of non-zero vorticity gives a *nearly* Hamiltonian structure (with a linear dependency on a vorticity term). Wahlén [19] then showed that, in fact, the system does indeed have *fully* Hamiltonian structure, which can be transformed into canonical form.

A consideration of a system consisting of 2 unbounded media with a free common interface was given by Benjamin and Bridges [20,21]. Craig et al. [22,23] considered an irrotational system consisting of a lower medium bound underneath by a flat bed and an upper medium bound above by an impermeable lid such that the 2 media have a free common interface and also the case in which the upper media itself has a free surface. The aim of this paper is to show that, in the rotational case, the 2 media bounded system has canonical Hamiltonian structure.

*E-mail address:* alan.compelli@mydit.ie.

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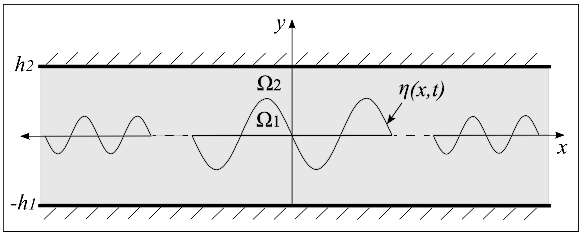


**Fig. 1.** The system under study.

## 2. Preliminaries

As per Fig. 1 we define the lower medium $\Omega_1$ as the domain $\{(x, y) \in \mathbb{R}^2 : -h_1 < y < \eta(x, t)\}$, the upper medium $\Omega_2$ as the domain $\{(x, y) \in \mathbb{R}^2 : \eta(x, t) < y < h_2\}$ and the entire system $\Omega$ as the domain $\{(x, y) \in \mathbb{R}^2 : -h_1 < y < h_2\}$ where $\{y = \eta(x, t)\}$ describes the elevation of the common interface. The subscript $c$ will be used to denote evaluation at the common interface.

We use the subscript notation $i = \{1, 2\}$ to represent the lower and upper media respectively and thus can consider a velocity potential $\varphi_i$ which is defined by:

$$\begin{cases} u_i = \partial_x \varphi_i - \omega_i y \\ v_i = \partial_y \varphi_i \end{cases} \tag{1}$$

where non-lateral velocity flow, with propagation in the positive $x$-direction, is given by $\mathbf{V_i}(x, y, z) = (u_i, v_i, 0)$ and $\omega_1$ and $\omega_2$ are the respective non-zero constant vorticities.

Additionally, the stream function $\psi_i$ is introduced, defined by:

$$\begin{cases} u_i = -\partial_y \psi_i \\ v_i = \partial_x \psi_i. \end{cases} \tag{2}$$

$\rho_1$ and $\rho_2$ are the respective constant densities of the lower and upper media and stability is given by the condition that $\rho_1 > \rho_2$.

We assume that for large $|x|$ the amplitude of $\eta$ attenuates and hence make the following assumptions

$$\lim_{|x|\to\infty} \eta(x, t) = 0, \tag{3}$$

$$\lim_{|x|\to\infty} \varphi_i(x, y, t) = 0, \tag{4}$$

and

$$-h_1 < \eta(x, t) < h_2 \quad \text{for all } x \text{ and } t. \tag{5}$$

## 3. Governing equations

We write Euler's momentum-conserving equation as:

$$\partial_t \mathbf{V_i} + (\mathbf{V_i}.\nabla)\mathbf{V_i} = -\frac{1}{\rho_i}\nabla P_i + \mathbf{g} \tag{6}$$

where $P_i = \rho_i g y + p_{\text{atm}} + p_i$ is the pressure at a depth $y$, $p_{\text{atm}}$ is (constant) atmospheric pressure, $p_i$ is the dynamic pressure due to the wave motion, $g$ is the acceleration due to gravity (where $y$ points in the opposite direction to the center of gravity) and $\mathbf{g}$ is the force due to gravity per unit mass.

Applying Eqs. (1) and (2) this can be written as

$$\nabla\Big(\partial_t \varphi_i + \frac{1}{2}(\nabla \psi_i)^2 - \omega_i \psi_i\Big) = \nabla\Big(-gy - \frac{p_i}{\rho_i}\Big) \tag{7}$$

where $\nabla = (\partial_x, \partial_y)$.

At the interface $p_1 = p_2 = p_c$ therefore we write Euler's equation in terms of the velocity potentials, stream functions, densities and vorticities as the energy conserving equality

$$\rho_1 \nabla\Big((\partial_t \varphi_1)_c + \frac{1}{2}(\nabla \psi_1)_c^2 - \omega_1 \chi_1 + g\eta\Big) - \rho_2 \nabla\Big((\partial_t \varphi_2)_c + \frac{1}{2}(\nabla \psi_2)_c^2 - \omega_2 \chi_2 + g\eta\Big) = 0, \tag{8}$$

where $\chi_i$ is the stream function evaluated at the interface.