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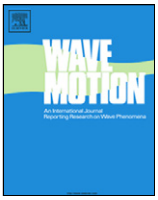

The effect of rotation and initial stress on the propagation of waves in a transversely isotropic elastic solid

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HIGHLIGHTS

- The effect of rotation on elastic wave propagation.
- The effect of initial stress on elastic wave propagation.
- Elastic wave propagation under combined initial stress and rotation.

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ABSTRACT

In this paper the equations governing small amplitude motions in a rotating transversely isotropic initially stressed elastic solid are derived, both for compressible and incompressible linearly elastic materials. The equations are first applied to study the effects of initial stress and rotation on the speed of homogeneous plane waves propagating in a configuration with uniform initial stress. The general forms of the constitutive law, stresses and the elasticity tensor are derived within the finite deformation context and then summarized for the considered transversely isotropic material with initial stress in terms of invariants, following which they are specialized for linear elastic response and, for an incompressible material, to the case of plane strain, which involves considerable simplification. The equations for two-dimensional motions in the considered plane are then applied to the study of Rayleigh waves in a rotating half-space with the initial stress parallel to its boundary and the preferred direction of transverse isotropy either parallel to or normal to the boundary within the sagittal plane. The secular equation governing the wave speed is then derived for a general strain–energy function in the plane strain specialization, which involves only two material parameters. The results are illustrated graphically, first by showing how the wave speed depends on the material parameters and the rotation without specifying the constitutive law and, second, for a simple material model to highlight the effects of the rotation and initial stress on the surface wave speed.

1. Introduction

In a recent paper [1] the equations governing small incremental motions superimposed on a deformed transversely isotropic initially stressed elastic solid were derived for both compressible and incompressible materials, the transverse isotropy being associated with a preferred direction in the initially stressed reference configuration of the material. This is

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in part based on an elastic strain–energy function that involves combined invariants of the right Cauchy–Green deformation tensor, the initial stress tensor and the preferred direction. For a compressible material there are 18 such independent invariants in three dimensions and 17 for an incompressible material. The theory was applied to the propagation of homogeneous plane waves with a restricted set of invariants, and, in particular, it was shown how the wave speed depends in a nonlinear fashion on the initial stress. Ogden and Singh [1] also studied Rayleigh surface waves in a half-space subject to initial stress parallel to its boundary with the preferred direction of transverse isotropy either parallel to or normal to the boundary, and they highlighted significant differences between the predictions of the general theory and of Biot's classical theory [2]; see also [3].

We refer to [1] for detailed references to the literature that deals with initially stressed materials and surface wave problems in the context of the linear theory of elasticity. For problems based on the nonlinear theory of elasticity, where the surface waves are propagating in a deformed and pre-stressed material we refer to, for example, [4–10]. See also the paper by Destrade and Saccomandi [11], which highlights an issue related to the effect of rotation in the finite deformation context.

The dependence of the characteristics of waves propagating in solids upon various geometrical and physical parameters remains a very active field of study. The effects of pre-stress, acceleration, and temperature variation, etc., on the wave speed or frequency are not only of theoretical interest but also provide the foundation for the development of many acoustic sensors [12]. In particular, frequency shifts due to rotation have been used to design angular rate sensors (see, for example, [13–15]).

A theoretical study of surface waves in a rotating half-space was first conducted by Schoenberg and Censor [16] for linearly isotropic elastic materials. They pointed out that bulk and surface waves are dispersive, and that the acoustic tensor is Hermitian, but they did not derive the secular equation; see also the paper by Auriault [17], who considered the effect of rotation on the speed of bulk waves in an isotropic linearly elastic solid. Clarke and Burdess [18] considered the rotation of an isotropic elastic half-space about an axis normal to the sagittal plane in the case of a small rate of rotation, and later extended their results for an arbitrary rotation rate [19]; see also [20,21], and, with particular reference to porous materials, the recent paper by Tomar and Ogden [22] and references therein.

For linear anisotropic materials, Destrade [23] presented an explicit secular equation for monoclinic materials with rotation about an axis perpendicular to a symmetry plane, and Destrade [24] also investigated the problem for orthotropic materials whose symmetry planes coincide with the coordinate planes and he obtained explicit secular equations for rotation about each of the symmetry axes. Ting [25] used the Stroh formalism to study surface waves in a rotating anisotropic elastic half-space. Wauer [26], Fang et al. [27] and Zhou and Jiang [28] studied surface waves in a rotating piezoelectric half-space; see also the review by Yang [29].

In the present paper we analyze the effect of rotation and initial stress on the propagation of plane and surface waves based on a general formulation of the constitutive law of a transversely isotropic initially stressed elastic material of Ogden and Singh [1]. In Section 2, the equations governing small amplitude incremental motions in a deformed rotating general elastic solid with initial stress are derived, both for compressible and incompressible materials, the transverse isotropy being associated with a preferred direction in the initially stressed configuration. To avoid complications of inhomogeneity of the underlying deformation associated with a steady rotation we then specialize to the linear theory of elasticity with uniform elastic moduli. In Section 3, the equations of motion are used to study the effects of initial stress and rotation on the speed of homogeneous plane waves. Formulas for the wave speed are obtained for a general form of constitutive law, for both compressible and incompressible materials, thus showing how the wave speed is adjusted by the rotation compared with the situation without rotation.

In Section 4 the constitutive law for a transversely isotropic initially stressed material is derived and expressions for the Cauchy and nominal stress tensors and the elasticity tensor are given. We then specialize to plane strain, which enables considerable simplification in that the number of invariants required is reduced from 18 to seven for a compressible material and 17 to six for an incompressible material. The equations governing incremental motions are also specialized to two-dimensional motions (specifically for incompressible materials) in the plane of the considered plane strain, in Section 5, again for linear elasticity, and applied to a study of Rayleigh waves in a half-space subject to initial stress parallel to its boundary with the preferred direction of transverse isotropy either parallel or normal to the boundary and with the rotation axis normal to the (sagittal) plane. The secular equation is derived in explicit form for a general transversely isotropic, initially stressed linearly elastic constitutive law within the plane strain specialization. This involves only two parameters that depend on the material properties, and we demonstrate the dependence on these parameters numerically without specifying the form of constitutive law. Finally, we provide numerical results in respect of a simple constitutive model that demonstrate the effect of the rotation and initial stress on the surface wave speed, and show how the results relate to those arising in the case of a constitutive law based on the classical theory of Biot [3].

2. Basic equations

Let $\mathcal{B}_r$ denote a fixed reference configuration of a material body within which a generic material point is denoted by its position vector $\mathbf{X}$. Let there be a stress distribution in this configuration, with the (Cauchy) stress tensor denoted $\boldsymbol{\tau}$. If there are no body forces and no intrinsic couples, which we assume to be the case in this paper, then $\boldsymbol{\tau}$ is symmetric and satisfies