Wave Motion 51 (2014) 1161–1169

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Shear-wave resonances in a fluid–solid–solid layered structure

P.A. Martin

Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, CO 80401-1887, USA

HIGHLIGHTS

- Properties of inhomogeneous layer modeled with exponentials.
- Exact solutions found using hypergeometric functions.
- Accurate asymptotic approximations developed.

ARTICLE INFO

Article history:
Received 17 December 2013
Accepted 4 July 2014
Available online 21 July 2014

Keywords:
Layered media
Hypergeometric function
Fluid–solid interaction

ABSTRACT

An inhomogeneous solid layer is bounded on one side by a fluid half-space and on the other by a homogeneous solid half-space. An acoustic wave in the fluid is incident on the layer. Experiments suggest that some kind of shear-wave resonance of the layer exists. Here, the layer is modeled with exponential variations of the material properties (Epstein model). Solutions in terms of hypergeometric functions are found. Genuine resonances are found but only when the layer is not bonded to the solid half-space; these are analogous to Jones frequencies in fluid–solid interaction problems. When the solid half-space is present, the resonances become complex: they are scattering frequencies. Simple but accurate asymptotic approximations are found using known estimates for hypergeometric functions with large parameters.

1. Introduction

Shear waves in marine sediments is the title of a 600-page edited book, published in 1991 [1]. Its subject has long been of interest to underwater acousticians. The basic model considered is a fluid (the ocean) on top of an inhomogeneous solid layer (the sediment) on top of a homogeneous solid (the basement). Such configurations (usually without the fluid) have also been studied in the context of seismology and soil dynamics.

Our motivation comes from studies by Godin and Chapman [2,3], and others, which show some kind of resonance behavior, attributed to shear waves in the inhomogeneous layer; see, especially, [2, Fig. 1]. In fact, these are not genuine resonance frequencies; they are complex scattering frequencies close to the real-frequency axis. We shall investigate these scattering frequencies, using mainly analytical methods.

Various analytical formulas have been used to represent the variations of the material properties through the inhomogeneous layer. For an isotropic elastic solid, lying between planes $z = 0$ and $z = h$, the relevant quantities are the Lamé moduli, $\lambda(z)$ and $\mu(z)$, and the density, $\rho(z)$. We shall assume exponential variations, giving models of Epstein type: in 1930, Epstein [4] considered acoustic waves in a continuously varying medium (not a layer), and he gave solutions in terms of hypergeometric functions; we shall encounter such functions later. For elastodynamic problems with Epstein models,

E-mail address: pamartin@mines.edu.

http://dx.doi.org/10.1016/j.wavemoti.2014.07.002
0165-2125/

see, for example, Rao [5], Vrettos [6], Muravskii [7], Rao and Li [8] and Manolis et al. [9]; all these authors assume that ρ and λ/μ are constants. The same assumptions are made by Godin and Chapman [2,3], but they use power-law variations of $\mu(z)$. Robins [10] allows ρ to vary but not the Lamé moduli. For two other approaches, see [11,12]. For more details on acoustic models, see [13, Chapter 3].

We begin by recalling the governing equations for the fluid–solid–solid problem. We consider two-dimensional motions, with plane-strain conditions in the solid regions. There is a plane time-harmonic acoustic wave in the fluid, incident upon the fluid–solid interface. Our focus is on normal incidence because then the whole problem decouples into two subproblems, one involving the acoustic pressure and the z-component of the elastic displacement (we call this the "compressional problem"), and one involving the other component of the displacement ("shear problem"). If the shear problem has any non-trivial solutions, such solutions do not couple to the fluid, and so the potential for resonance would arise. Indeed, such real resonance frequencies do exist but only when the layer is not bonded to the homogeneous elastic half-space. (This is a simple consequence of Sturm–Liouville theory.) When the layer is bonded to the half-space, we find complex scattering frequencies. In both cases, the frequencies are found by setting an appropriate 2×2 determinant to zero. We also give a brief discussion of a semi-infinite smoothly inhomogeneous half-space (so that there is no interface at $z = h$).

The numerical results are compared with simple asymptotic approximations. These are obtained by approximating the relevant determinants, which consist of products of hypergeometric functions. Unusually, we have to estimate such functions when their argument is fixed but their parameters are large; for example, $F(1+\delta, 1+\delta; 1+2\delta; \zeta)$ when ζ is fixed but $\delta \to \infty$. Fortunately, an appropriate (but complicated) asymptotic approximation was given by G.N. Watson almost 100 years ago. (For a recent review of this topic, see [14].) It turns out that the asymptotic approximations give excellent agreement with the numerical results.

2. Formulation of the problem

We consider a three-part layered medium with two flat interfaces, at $z = 0$ and $z = h > 0$.

In the semi-infinite region $z < 0$ (the "water"), there is a homogeneous compressible inviscid fluid with density ρ_{f} and sound speed c_{f}. The pressure P satisfies the wave equation for $z < 0$.

In the semi-infinite region $z > h$ (the "substrate"), there is a homogeneous isotropic elastic solid with density ρ_{s} and Lamé moduli μ_{s} and λ_{s}.

In the region $0 < z < h$ (the "layer"), there is an inhomogeneous isotropic elastic solid with density $\rho(z)$ and Lamé moduli $\mu(z)$ and $\lambda(z)$.

For plane-strain motions in the solid regions, the elastodynamic displacement vector has components $u_1(x, z, t)$ and $u_3(x, z, t)$ in the x and z directions, respectively. The governing equations are

$$\rho\frac{\partial^2 u_1}{\partial t^2} = (\lambda+2\mu)\frac{\partial^2 u_1}{\partial x^2} + \mu\frac{\partial^2 u_1}{\partial z^2} + (\lambda+\mu)\frac{\partial^2 u_3}{\partial x\,\partial z} + \mu'(z)\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right), \tag{1}$$

$$\rho\frac{\partial^2 u_3}{\partial t^2} = (\lambda+2\mu)\frac{\partial^2 u_3}{\partial z^2} + \mu\frac{\partial^2 u_3}{\partial x^2} + (\lambda+\mu)\frac{\partial^2 u_1}{\partial x\,\partial z} + \lambda'(z)\frac{\partial u_1}{\partial x} + (\lambda'+2\mu')\frac{\partial u_3}{\partial z}. \tag{2}$$

The relevant stresses are

$$\sigma_{33} = \lambda\frac{\partial u_1}{\partial x} + (\lambda+2\mu)\frac{\partial u_3}{\partial z}, \qquad \sigma_{13} = \mu\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right). \tag{3}$$

At the water-layer interface, the boundary conditions are

$$\frac{\partial P}{\partial z} + \rho_{\mathrm{f}}\frac{\partial^2 u_3}{\partial t^2} = 0, \qquad P + \sigma_{33} = 0 \quad \text{and} \quad \sigma_{13} = 0 \quad \text{at } z = 0. \tag{4}$$

At the layer-substrate interface, the boundary conditions are

$$u_1,\ u_3,\ \sigma_{13} \text{ and } \sigma_{33} \text{ are continuous across } z = h. \tag{5}$$

2.1. Time-harmonic motions

Suppose now that

$$P(x,z,t) = p(z)\,\mathrm{e}^{\mathrm{i}(\xi x-\omega t)}, \qquad u_1(x,z,t) = \mathrm{i}\,u(z)\,\mathrm{e}^{\mathrm{i}(\xi x-\omega t)}, \qquad u_3(x,z,t) = w(z)\,\mathrm{e}^{\mathrm{i}(\xi x-\omega t)}.$$

(The factor i in u_1 is inserted for algebraic convenience.) In the water, we have

$$p''(z) + \{(\omega/c_{\mathrm{f}})^2 - \xi^2\}p(z) = 0.$$

In the solid regions, Eqs. (1) and (2) become

$$\mu u'' + \mu' u' + [\rho\omega^2 - (\lambda+2\mu)\xi^2]u + (\lambda+\mu)\xi w' + \mu'\xi w = 0, \tag{6}$$

$$(\lambda+2\mu)w'' + (\lambda'+2\mu')w' + (\rho\omega^2 - \mu\xi^2)w - (\lambda+\mu)\xi u' - \lambda'\xi u = 0. \tag{7}$$