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A facile method to realize perfectly matched layers for elastic waves

Zheng Chang [a,b], Dengke Guo [a], Xi-Qiao Feng [b], Gengkai Hu [a,*]

[a] Key Laboratory of Dynamics and Control of Flight Vehicle, Ministry of Education, School of Aerospace Engineering, Beijing Institute of Technology, Beijing 100081, China

[b] CNMM & AML, Department of Engineering Mechanics, Tsinghua University, Beijing 100084, China

HIGHLIGHTS

- The design method of PMLs is proposed based on transformation elastodynamics.
- The PML can be easily realized by functionally graded viscoelastic materials.
- Numerical simulations are performed to validate the performance of the PMLs.

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ABSTRACT

In perfectly matched layer (PML) technique, an artificial layer is introduced in the simulation of wave propagation as a boundary condition which absorbs all incident waves without any reflection. Such a layer is generally thought to be unrealizable due to its complicated material formulation. In this paper, on the basis of transformation elastodynamics and complex coordinate transformation, a novel method is proposed to design PMLs for elastic waves. By applying the conformal transformation technique, the proposed PML is formulated in terms of conventional constitutive parameters and then can be easily realized by functionally graded viscoelastic materials. We perform numerical simulations to validate the material realization and performance of this PML.

1. Introduction

In 1994, Berenger [1] proposed a perfectly matched layer (PML) technique to improve the efficiency of wave propagation simulations. A PML is an artificial layer introduced as boundary conditions in calculation model, which absorbs all incident waves without any reflection. The key strategy in this technique is to perform a coordinate transformation which maps the real harmonic wave $e^{i\mathbf{k}\cdot\mathbf{x}}$ in the real space $\mathbf{x}$ into the following form in a complex space $\hat{\mathbf{x}} = \mathbf{x}' + i\mathbf{x}''$:

$$e^{i\mathbf{k}\cdot\hat{\mathbf{x}}} = e^{-\mathbf{k}\cdot\mathbf{x}''}e^{i\mathbf{k}\cdot\mathbf{x}'}, \tag{1}$$

where $\mathbf{k}$ is the wavevector. Thus the coordinate transformation makes the wave attenuate in the exponential manner $e^{-\mathbf{k}\cdot\mathbf{x}''}$. In the early understanding, the PMLs were thought to be "nonphysical" and "purely mathematical". Later, an alternative formulation of PMLs in electrodynamics was reported [2,3], known as "Maxwellian PMLs", in which the coordinate deformation

* Corresponding author. Tel.: +86 10 68918363; fax: +86 10 68914538.
E-mail address: hugeng@bit.edu.cn (G. Hu).

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is described by complex material parameters. These works not only revealed the physical interpretation of PMLs but also showed the possibility of their realization in practical materials.

Coincidentally, the similar idea of equivalence between material distribution and space distortion has been employed in the transformation method (TM) [4,5], or more specifically, the transformation optics (TO) in the field of electrodynamics, to design novel wave functional devices, such as "invisibility cloak" [6]. Teixeira et al. [7] investigated the relation between the PML method and the TM method. They demonstrated that the complex transformation in the former method can be regarded as a generalization of the latter. It is also noticed that in the field of elastodynamics, there is a similar coincidence. The material formulation of elastodynamic PMLs [8] has the same form as that derived by transformation elastodynamics [9] with the stiffness tensor without minor symmetry. Recently, Chang et al. [10,11] designed a simplified form of elastodynamic PMLs with a symmetric elastic tensor on the basis of an approximate transformation elastodynamic method. This type of PMLs have been proved by a number of numerical examples to have good absorption property [10,11], but it remains unclear how they can be realized in practice. In addition, the TM has been demonstrated as a convenient tool to elucidate the physical insight of PMLs and to ease the design process of PMLs. For instance, Popa et al. [12] used this technique to design an electromagnetic PML of arbitrary shape in the context of transformation optics. In the present study, the TM will be used to design elastodynamic PMLs and to simplify the designed PMLs to a realizable level.

As PMLs are mathematically formulated in a complex space, their realization by real materials is a technologically important issue. In elastodynamics, a good absorption ability to incident wave is of great interest in both experimental and engineering applications. Especially, if there is no effective absorption mechanism, elastic waves could not be measured in steady state in elastodynamic experiments. Even in transient measurements, the sample has to be very large in order to avoid the interference of reflected waves from boundaries. For a long time, the PML technique has not been realized in practice but only used in numerical simulations. The main obstacle to realize the PML technique is that the mathematically derived PMLs require asymmetric elastic tensors [8], which do not exist in nature. This difficulty will be overcome in the present paper by applying conformal transformation technique.

In this study, a design method of an elastodynamic PML is proposed on the basis of transformation elastodynamics. We will show that an elastodynamic PML can be formulated in terms of conventional material parameters and can be fabricated with conventional viscoelastic materials. The paper is organized as follows. In Section 2, an elastodynamic PML is proposed on the basis of transformation elastodynamics and validated via a number of numerical simulations. In Section 3, the formulation of the proposed PML is further simplified to make them physically realizable. Especially, a simple prototype of PMLs is given. Finally, the main results are summarized in Section 4.

2. Elastodynamic PML

In this section, we first derive the formulation of elastodynamic PMLs by using the TM. The governing equation of elastic waves, Navier's equation, is written as [10]

$$\nabla \cdot \mathbf{C} : \nabla \mathbf{u} = -\omega^2 \boldsymbol{\rho} \cdot \mathbf{u}, \tag{2}$$

where $\mathbf{C}$ is the fourth-order elastic tensor, $\mathbf{u}$ is the wave displacement vector, $\boldsymbol{\rho}$ is the anisotropic mass density tensor, and ω is the angular frequency. As the core of transformation elastodynamics, the form invariance of Eq. (2) under coordinate transformation has been investigated by various methods [9–11,13,14]. Milton et al. [13] first raised the problem and showed by so-called "change of variable" approach that Navier's equation in (2) will not preserve its original form but will be transformed into Willis' equation under the prescribed transformation relation of displacement

$$\bar{\mathbf{u}} = \mathbf{A}^{-T} \mathbf{u}, \tag{3}$$

where $A_{Nn} = \partial x_N / \partial X_n$ is the Jacobian matrix of space (or coordinate) transformation. This implies the transformation method can only be available under the condition that the transformed material obeys Willis equation. Later, Brun et al. [9] demonstrated a precise control of elastic wave by using transformation method with the following transformation relation of displacement vector

$$\bar{\mathbf{u}} = \mathbf{I}\mathbf{u}, \tag{4}$$

where $I_{Ii} = \delta_{Ii}$ is the second-order unit tensor. In this method, the transformed material is of Navier's form, but the asymmetric elastic tensor is needed to guarantee the form invariance. Furthermore, Norris and Shuvalov [14] developed a general elastic cloaking theory in which a "gauge" matrix is applied such that both above two works are particular cases of the theory. On the other hand, Chang et al. [10] proposed an alternative method to obtain possible transformation relations which can keep the form of governing equations (e.g. Maxwell equations for electromagnetic waves and Helmholtz equation for acoustic waves) during coordinate transformation. They found for elastodynamics, the form invariance of Eq. (2) can be approximately preserved in case where the gradient of the elastic moduli of the transformed material is small or when wave frequency is high [11].

By introducing complex coordinate transformation, the above methods can all be applied to accomplish the design of PMLs. However, for the ease of numerical implementation and further simplification, only Navier's form of transformed material is considered in this work.