Wave Motion 51 (2014) 1237–1253

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Precursors for waves in random media

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HIGHLIGHTS

- We model propagation of 3D waves in random media in the paraxial approximation.
- We obtain a generalized O'Doherty–Anstey theory.
- We validate the results numerically.

ARTICLE INFO

Article history:
Received 16 August 2013
Received in revised form 1 July 2014
Accepted 18 July 2014
Available online 24 July 2014

Keywords:
Parabolic approximation
Random media
Precursor
O'Doherty–Anstey approximation

ABSTRACT

We consider scattering of a pulse propagating through a three-dimensional random media and study the shape of the pulse in the parabolic approximation. We show that, similarly to the one-dimensional O'Doherty–Anstey theory, the pulse undergoes a deterministic broadening. Its amplitude decays only algebraically and not exponentially in time, due to the signal low/midrange frequency component. We also argue that the parabolic approximation captures the front evolution (but not the signal away from the front) correctly even in the fully three-dimensional situation.

1. Introduction

The problem of imaging in heterogeneous environments arises in many important applications such as biomedical imaging, telecommunications, seismic exploration in geophysics or non-destructive testing of materials. One standard technique consists in probing a medium with e.g. an electromagnetic pulse and by collecting the echos on an array of detectors. How well the method will perform strongly depends on the structure of the wavefield that propagates in the complex medium. In particular, if the target to be imaged is buried deeply into the medium, scattering effects are important and the measured wavefield might not be strong enough to be used in the inversion.

The main objectives of this work are to determine the spreading and decay of the pulse amplitude in the medium, and to characterize the optimal frequency content of the source in order to probe at a given depth. In [1] the authors showed that a pulse traveling in a layered or one-dimensional medium that fluctuates randomly on a fine scale is affected by the microstructure in a particular way: the pulse shape undergoes a transformation that can be described in terms of a convolution with a pulse shaping function that depends on the statistics of the microstructure, while its travel time to a

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http://dx.doi.org/10.1016/j.wavemoti.2014.07.008

specific depth has a small random component that depends on the particular realization of the microstructure. This has come to be known as the O'Doherty–Anstey approximation. A string of papers has subsequently considered this approximation and its generalizations and applications. In [2] the mathematical theory for this theory in the layered case is presented in detail, see also [3–6] for early references. The approximation was considered for the so called locally layered media in [7]. In recent works, see for instance [8,9], the situation with propagation in so called long range or fractional media is considered. These works discuss the link in between the roughness and also slow decay of correlations for the layered microstructure and the character of the associated O'Doherty–Anstey approximation. This link was used in [10] to describe the so called fractional precursors in wave propagation. The intuitive picture is that the pulse shaping kernel acts as a low pass filter and as the pulse travels deep into the medium the low frequencies that survive may appear as a pulse centered at lower frequencies and which exhibit only algebraic decay if the low frequency contents are rich enough. Eventually though, if the frequency support of the probing pulse is supported away from the origin, the pulse will exhibit exponential decay as the wave energy is converted to incoherent fluctuations due to the scattering.

This is to be compared with the classical Brillouin precursor that arises in dispersive, dissipative media [11]: when a broadband pulse propagates in a lossy dispersive medium, the combination of dispersive and absorption effects lead to the generation of a low-frequency precursor field which decays algebraically and not exponentially, see [12–16]. In our context, the medium of interest is non-dispersive and non-dissipative, and the scattering effects by the heterogeneous medium are responsible for the existence of the precursor. There is though some dispersion in our approach since we will describe the propagation in the paraxial approximation where the wavefield is solution to a dispersive equation. Brillouin precursors offer applications in remote sensing and communications for instance, as well as the study of materials and biological systems [17–20].

Here we ask the question if such a picture in fact generalizes to a two or three dimensional wave propagation scenario. Indeed we find that core elements of these results persist. Specifically we consider acoustic waves in two or three spatial dimensions and model the complex medium by a random medium with appropriate statistics. The propagation of the waves will be described in the parabolic approximation [21] where the complex amplitude of the pressure is solution to a stochastic Schrödinger equation. More precisely, we consider the scalar wave equation in a random medium

$$\frac{1}{c^2(\vec{X})}\frac{\partial^2 u}{\partial T^2} - \Delta u = 0, \quad T > 0, \vec{X} = (X, Z) \in \mathbb{R}^{d+1}, \tag{1}$$

with $d = 1, 2$ and the local wave speed $c(X, Z)$ of the form

$$c^{-2}(X, Z) = c_0^{-2}\left[1 + \sigma_0\mu\left(\frac{X}{l}, \frac{Z}{l}\right)\right].$$

Here, c_0 is a constant reference speed, $Z \in \mathbb{R}^+$ and $X \in \mathbb{R}^d$ are, respectively, the coordinates along and transverse to the direction of propagation. The random function μ models fluctuations with amplitude σ_0 and correlation length l in the propagation speed. Solutions of the wave equation (1) may be written in the form

$$u(T, X, Z) = \frac{1}{2\pi}\int e^{i\omega(Z/c_0 - T)}\psi\left(Z, X; \frac{\omega}{c_0}\right) d\omega, \tag{2}$$

where the complex amplitude $\psi(Z, X; k = K)$ satisfies the Helmholtz equation

$$2iK\frac{\partial\psi}{\partial Z} + \Delta_X\psi + K^2(n^2 - 1)\psi = -\frac{\partial^2\psi}{\partial Z}. \tag{3}$$

Here $K = \omega/c_0$ is the wavenumber and $n(X, Z) = c_0/c(X, Z)$ is the random index of refraction relative to c_0. The fluctuations of the refraction index have the form

$$n^2(X, Z) - 1 = \sigma_0\mu\left(\frac{X}{l}, \frac{Z}{l}\right).$$

The normalized and dimensionless covariance is given by

$$\tilde{R}(Z, X) = \mathbb{E}\{\mu(X + X', Z + Z')\mu(X', Z')\}.$$

We assume that the typical propagation distance in the Z-direction is L_z, the transverse variation (say, of the initial pulse profile) is L_x, and k_0 is a central wavenumber associated with our source. We obtain the dimensionless form of (3) by introducing the dimensionless variables $X = L_x x$, $Z = L_z z$, $K = k_0 k$ and rewriting the equation as

$$2ik\left(\frac{1}{k_0 L_z}\right)\frac{\partial\psi}{\partial z} + \left(\frac{1}{k_0 L_x}\right)^2\Delta_x\psi + k^2\sigma_0\mu\left(\frac{zL_z}{l}, \frac{xL_x}{l}\right)\psi = -\frac{1}{(k_0 L_z)^2}\frac{\partial^2\psi}{\partial z^2}. \tag{4}$$

We assume now that the medium fluctuates on a relatively fine scale and accordingly introduce the small parameters $\varepsilon_x = l/L_x$ and $\varepsilon_z = l/L_z$. Then (4) takes the form

$$2ik\left(\frac{\varepsilon_z}{k_0 l}\right)\frac{\partial\psi}{\partial z} + \left(\frac{\varepsilon_x}{k_0 l}\right)^2\Delta_x\psi + k^2\sigma_0\mu\left(\frac{z}{\varepsilon_z}, \frac{x}{\varepsilon_x}\right)\psi = -\frac{\varepsilon_z^2}{(k_0 l)^2}\frac{\partial^2\psi}{\partial z^2}. \tag{5}$$