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On the steady solitary-wave solution of the Green–Naghdi equations of different levels

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HIGHLIGHTS

- Solution of the high-level GN equations are obtained for steady solitary waves.
- The classical nonlinear Green–Naghdi theories are solved numerically.
- The solutions at each level are studied for convergence and then compared with existing experimental data and other predictions.
- The agreement between the converged GN solution (trajectories, velocities and surface elevation) and experiments and Euler's equations is very good.

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ABSTRACT

The steady-state solitary wave solution of high-level Green–Naghdi (GN) equations is obtained by use of the Newton–Raphson method. Four aspects of solitary waves are studied: the wave speed, wave profile, velocity field and particle trajectory. A convergence study is performed for each individual case. Results of the converged model are compared with the existing laboratory experiments and other theoretical solutions for an inviscid and incompressible fluid, including the solutions of the Euler equations. Particle trajectories, predicted by the GN model, show close agreement with the laboratory measurements and provide a new approach to understanding the movement of the particles under a solitary wave. It is further shown that high-level GN equations can predict the solitary wave of the highest height.

1. Introduction

A solitary wave is known to be the infinite wave length limit of a cnoidal wave. Many studies have focused on the solitary wave, such as by Korteweg and de Vries [1], Daily and Stephan [2], and Fenton [3]. Longuet-Higgins [4] conducted some experiments on trajectories of particles on the surface of steep solitary waves. Experimental data and theoretical calculations of Longuet-Higgins [4] show that the horizontal displacement of particles on the surface of steep solitary waves exceeds the prediction of the Korteweg–de Vries equation, by as much as 100%.

During the past two decades and with the advances made in visualization techniques, a new window is opened into the research of complex fluid flows. The developments in particle image velocimetry (PIV), for instance, have led to the

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visualization of velocity fields and particle trajectories with great accuracies. More recent experiments on solitary waves can be found in, e.g., [5–7].

Theoretical investigations of particle trajectories under a solitary wave were also undertaken by Constantin and Escher [8] and Constantin [9]. They, however, did not give any numerical results on particle trajectories. Some numerical work has been done based on the KdV theory (see [10,7]). Umeyama [7] shows that the KdV theory underpredicts the horizontal displacement of the particle compared with his laboratory measurements (see Fig. 4 in his paper). Therefore, it is of interest to find an accurate model to predict the particle velocity and trajectory under a solitary wave.

In this study, we shall use a direct approach, namely the Green–Naghdi theory of water waves, originally developed by Green and Naghdi [11]. We will study the numerical solution of the velocity field and particle trajectory under a solitary wave by solving the Green–Naghdi (GN, hereafter) equations. The governing equations in the GN model are the depth-integrated form of the Euler equations for an inviscid and incompressible fluid. In the original GN equations, both the bottom boundary condition and the nonlinear free surface boundary conditions are satisfied exactly. Distribution of the velocity field over the water column is prescribed by approximated functions. The GN models are classified into different levels, such as GN-1, GN-2, . . . , and so forth, based on these approximated functions. In the GN-1 model, for example, the prescribed horizontal velocity distribution in the vertical direction is constant, while it is a linear and quadratic polynomial in the GN-2 and GN-3 models, respectively.

Shields and Webster [12] discussed the solitary wave solution of the GN-1, GN-2 and GN-3 models. They found that the GN models converge more rapidly than the perturbation theories, such as the solution of Fenton [3]. Partially due to the complexities associated with the GN equations, analytical solution of the solitary wave is limited to GN-1 given by e.g., [13]. Webster et al. [14] simplified the GN equations, thus enabling the utility of high-level GN theories in an easier manner. More recently, high-level GN theory was applied to water wave simulations (see [15,16]).

The set of GN equations for any level derived by Webster et al. [14] are essentially the same equations as the original GN equations, however, with a simpler structure that can be calculated much more easily. A quick look at the original GN equations would show that at second and higher levels, the equations are very complicated, see e.g., [17]. The original GN equations satisfy the conservation equations in the depth integrated sense exactly, and the boundary conditions on the free surface and on rigid boundaries exactly, see e.g., [12,18]. The GN equations as used here do satisfy therefore all the conservation equations for an inviscid and incompressible fluid and the boundary conditions exactly. Also see the introduction section of Zhao et al. [16] for more discussion on the subject.

We should mention that the original GN equations are not restricted to irrotational flows, that is, the GN model can be used to analyze rotational flows as well. Kim et al. [19] derived the Irrotational Green–Naghdi (IGN) equations from Hamilton's principle. The IGN equations have also been derived for finite water depth conditions and numerically tested to study self-convergence and accuracy of the models [20,21]. Numerically-obtained solitary-wave solution of high-level IGN equation were given by Kim et al. [20], who studied celerity, mass and energy characteristics of a solitary wave, and by Zhao et al. [22] who study the particle trajectory under a solitary wave, predicted by the IGN equations.

Our main goal in this work is to examine the applicability and accuracy of high-level GN equations in studying the wave speed, wave profile, velocity distribution and particle trajectories under solitary waves. In particular, a numerical solution of the particle velocity distribution at different layers under a solitary wave is developed. In addition, we provide accurate solutions of particle trajectories under a solitary wave. To do these, we first obtain the converged solutions through a GN model. The high-level GN equations (such as the GN-4 equations) is needed to check the self-convergence of the GN models, as the first three levels of the GN models give three different results on the horizontal component of wave velocity. Here, the numerically-obtained steady-state solitary-wave solution of the GN equations are obtained by use of the Newton–Raphson method. Then, we provide a comparison of the converged GN results with the existing laboratory measurements of Hsu et al. [5,6] and Umeyama [7].

In Section 2, high-level GN models are introduced. Section 3 gives the method to obtain the steady solitary wave solution of the GN models. The approach on calculating the particle trajectories is given in Section 4. In the remaining part of the paper, we are mainly concerned with the presentation of the numerical test cases, discussion and concluding remarks.

2. The GN equations

In this 2-dimensional study, x is the horizontal and z is the vertical coordinate and the origin of the coordinate system is located at the still-water level. $z = \beta(x, t)$ specifies the free surface and $z = \alpha(x) = -d$ represents the sea-floor surface, where d is the constant water depth. The fluid is assumed to be incompressible and inviscid.

In the GN model [16], the velocity field is estimated as

$$u(x, z, t) = \sum_{n=0}^{K-1} u_n(x, t) z^n, \tag{1a}$$

$$w(x, z, t) = \sum_{n=0}^{K} w_n(x, t) z^n, \tag{1b}$$

where u_n and w_n are the unknown velocity coefficients.