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The electromagnetic fields and the radiation of a spatio-temporally varying electric current loop

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HIGHLIGHTS

- Electric and magnetic fields of a spatio-temporally varying electric current loop are calculated.
- Radiation and the nonradiation parts of the electromagnetic fields are derived.
- A new, exact, analytical solution of the Maxwell equation has been found.

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ABSTRACT

The electric and magnetic fields of a spatio-temporally varying electric current loop are calculated using the Jefimenko equations. The radiation and the nonradiation parts of the electromagnetic fields are derived in the framework of Maxwell's theory of electromagnetic fields. In this way, a new, exact, analytical solution of the Maxwell equation is found.

1. Introduction

A current loop is an important item in the theory of electromagnetic induction and magnetostatics [1–4]. Usually, such considerations are restricted to the static case or quasistatic case. No single work has considered the general case of the non-uniform motion of an arbitrary current loop or the electromagnetic radiation fields of such a loop in the general framework of Maxwell's theory of electromagnetic fields, until now.

One important feature of the theory of electrodynamics is the retardation which is a consequence of the finite speed of the propagating electromagnetic fields. There is always a time delay since an effect observed by the receiver at the present position and present time was caused by the sender at some earlier time (retarded time) and at the retarded position.

The aim of this paper is the study of a spatio-temporally varying (or time-depending) closed current loop and the corresponding radiation fields. In particular we investigate the so-called electrokinetic field. The exact solutions of all the electromagnetic fields induced by a non-stationary electric current loop are determined. The Liénard–Wiechert fields produced by the electric current loop are calculated. The generalized Faraday law and the generalized Biot–Savart law for a spatio-temporally varying electric current loop are found. The electromagnetic fields are decomposed into radiation and

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nonradiation parts and finally, it is shown that the classical expression for an electric wire is recovered as static limit from our general expressions.

2. Basic framework

The basic electromagnetic field laws are represented by the inhomogeneous and homogeneous Maxwell equations [2,3]

$$\nabla \cdot \boldsymbol{D} = \rho, \qquad \nabla \times \boldsymbol{H} - \partial_t \boldsymbol{D} = \boldsymbol{J}, \tag{1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \times \boldsymbol{E} + \partial_t \boldsymbol{B} = 0, \tag{2}$$

where $\boldsymbol{D}$ is the electric displacement vector (electric excitation), $\boldsymbol{H}$ is the magnetic excitation vector, $\boldsymbol{B}$ is the magnetic field strength vector, $\boldsymbol{E}$ is the electric field strength vector, $\boldsymbol{J}$ is the electric current density vector, and ρ is the electric charge density. ∂_t denotes the differentiation with respect to the time t and ∇ is the Nabla operator. In addition, the electric current density vector and the electric charge density fulfill the continuity equation

$$\nabla \cdot \boldsymbol{J} + \partial_t \rho = 0. \tag{3}$$

The constitutive equations for the fields in a vacuum (Maxwell–Lorentz relations) read

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E}, \qquad \boldsymbol{H} = \frac{1}{\mu_0} \boldsymbol{B}, \tag{4}$$

where ϵ_0 is the vacuum permittivity and μ_0 is the vacuum permeability. The speed of light in vacuum is given by

$$c^2 = \frac{1}{\epsilon_0 \mu_0}. \tag{5}$$

From the Maxwell equations (1) and (2), inhomogeneous wave equations for the electromagnetic field strengths follow

$$\Box \boldsymbol{E} = -\frac{1}{\epsilon_0}\left(\nabla \rho + \frac{1}{c^2}\,\partial_t \boldsymbol{J}\right) \tag{6}$$

and

$$\Box \boldsymbol{B} = \mu_0 \nabla \times \boldsymbol{J}, \tag{7}$$

where the d'Alembert operator is defined by

$$\Box := \frac{1}{c^2}\,\partial_{tt} - \Delta \quad \text{with } \Delta = \nabla \cdot \nabla. \tag{8}$$

For zero initial conditions, using the retarded Green function of the wave equation and some mathematical manipulations, the causal solutions of the inhomogeneous wave equations (6) and (7) are given by retarded electromagnetic field strength vectors:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \left(\frac{\rho(\boldsymbol{r}', t - R/c)}{R^3}\,\boldsymbol{R} + \frac{\partial_t \rho(\boldsymbol{r}', t - R/c)}{cR^2}\,\boldsymbol{R} - \frac{\partial_t \boldsymbol{J}(\boldsymbol{r}', t - R/c)}{c^2 R} \right) \mathrm{d}\boldsymbol{r}', \tag{9}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \left(\frac{\boldsymbol{J}(\boldsymbol{r}', t - R/c)}{R^3} + \frac{\partial_t \boldsymbol{J}(\boldsymbol{r}', t - R/c)}{cR^2} \right) \times \boldsymbol{R}\, \mathrm{d}\boldsymbol{r}', \tag{10}$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$, $R = |\boldsymbol{r} - \boldsymbol{r}'|$, $\boldsymbol{r} \in \mathbb{R}^3$, $t \in \mathbb{R}$ and $\mathcal{V}$ denotes the whole 3-dimensional space. Eq. (9) is the time-dependent generalized Coulomb–Faraday law and Eq. (10) is the time-dependent generalized Biot–Savart law (see also [4]). Eqs. (9) and (10) express the electromagnetic fields in terms of their retarded sources $\rho, \boldsymbol{J}, \partial_t \rho$ and $\partial_t \boldsymbol{J}$ with full generality. They were originally derived by Jefimenko [5] (see also [6,7]). They appear also in the book of Clemmow and Dougherty [8] and in the third edition of Lorrain, Corson and Lorrain [9]. An equivalent representation was given by Panofsky and Phillips [1] (see also [7]). Both equations are nowadays called the Jefimenko equations in standard books on electrodynamics (e.g. [2–4]). They are fundamental, elegant and very useful equations.

3. A spatio-temporally varying electric current loop

We investigate a spatio-temporally varying electric current loop. We consider a closed loop of arbitrary shape (planar or non-planar) that can move arbitrary. The current density vector of a time-variable or spatio-temporally varying electric current loop at the time-dependent position $\boldsymbol{s}(t)$ is given by a line integral of the form

$$\boldsymbol{J}(\boldsymbol{r}, t) = \oint_{L(t)} I(t)\, \delta(\boldsymbol{r} - \boldsymbol{s}(t))\, \mathrm{d}\boldsymbol{L}(\boldsymbol{s}(t)), \tag{11}$$