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A hierarchy of dynamic equations for solid isotropic circular cylinders

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HIGHLIGHTS

- Variationally consistent cylinder equations for longitudinal, torsional and flexural modes.
- Series expansion of 3D theory, rendering a hierarchy of equations that are asymptotically correct.
- Analytical expressions for cylinder equations truncated to different orders.
- Numerical results for dispersion curves, eigenfrequencies, and forced motions.

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ABSTRACT

This work considers homogeneous isotropic circular cylinders adopting a power series expansion method in the radial coordinate. Equations of motion together with consistent sets of end boundary conditions are derived in a systematic fashion up to arbitrary order using a generalized Hamilton's principle. Time domain partial differential equations are obtained for longitudinal, torsional, and flexural modes, where these equations are asymptotically correct to all studied orders. Numerical examples are presented for different sorts of problems, using exact theory, the present series expansion theories of different order, and various classical theories. These results cover dispersion curves, eigenfrequencies and the corresponding displacement and stress distributions, as well as fix frequency motion due to prescribed end displacement or lateral distributed forces. The results illustrate that the present approach may render benchmark solutions provided higher order truncations are used, and act as engineering cylinder equations using low order truncation.

1. Introduction

Dynamic equations of circular cylinders have been studied extensively by many authors, from the full three dimensional geometry to simple one-dimensional models. Exact solutions to the three dimensional equations were derived by Pochhammer [1] and Chree [2] for infinitely long cylinders, resulting in the well-known transcendental frequency equations. Further investigations of these frequency equations have been carried out for the various mode families (longitudinal, torsional, and flexural). Studies of the corresponding dispersion curves have been investigated in detail [3–5] where results have been presented for the modes that are of most importance to the present work: the transverse lowest flexural modes and the higher order flexural mode families. Considering semi-infinite and finite cylinders, analytical transient solutions have been developed for the flexural modes in the special case of mixed boundary conditions [6,7]. Due to the complexity of the three dimensional theory of elastodynamics in general, and dealing with other end boundary conditions in particular (e.g. Dirichlet or Neumann), various approximate methods of solutions have been developed. There exists on one hand

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analytical solutions based on expansion in terms of Bessel functions [8,9] where part of the boundary conditions is satisfied approximately, and on the other hand numerical solutions such as the Ritz method [10–13] or the finite element method [14]. These works using approximate methods concentrate mainly on eigenfrequency analyses.

However, the bulk of analysis on flexural problems has been directed towards the lowest transverse flexural mode family using simplified one-dimensional beam theories. In these simplified theories, both the dynamic equations and the boundary conditions are often derived using various kinds of simplifying kinematic assumptions. The most used approximate theory is the simple Euler–Bernoulli equation, where shear and rotary inertia are neglected. This leads to a differential equation that has the undesired feature of being non-hyperbolic. However, if the beam radius is much smaller than both the beam length and the wavelengths, this approximation is known to yield accurate results. The next level of models include shear and rotary inertia described by Timoshenko [15], resulting in a hyperbolic equation of motion. There are several other more advanced beam theories in use. Some of these concern only rectangular cross sections [16–23], while others are applicable for circular cross sections [24–29].

The present paper aims at systematically develop a hierarchy of approximate equations for solid isotropic circular cylinders. To this end power series expansions in the radial coordinate are adopted in the three dimensional equations of motion. Using generalized Hamilton's principle, time domain equations of motions together with general lateral and end boundary conditions are stated in a systematic manner. Sets of cylinder equations may hereby be derived to an (in principle) arbitrary order for the various displacement families (longitudinal, torsional, flexural), where each studied truncated order is asymptotically correct. Higher order sets of time domain equations may be used for benchmark solutions to various three dimensional cylinder problems, while lower order sets may be used as alternative engineering equations. As the longitudinal axisymmetric motion is investigated in a separate paper by Folkow and Mauritsson [30], the main contribution here concerns torsional and the more involved flexural motions. Especially the lowest flexural transverse modes are studied in some detail.

Other higher order power series expansion flexural theories are presented in the literature [20,24–26,28]. Here, Martin [28] uses an approach on cylindrically anisotropic cylinders that in many respects is similar to the present method. The other cited works use approaches that are different from the present one concerning the series expansion method, the use of recursion relations, the procedure when collecting terms or the truncation process as a whole. Consequently, the resulting flexural theories (equations of motion, boundary conditions) are all different, even for the lowest truncation order. Besides solid cylinders, the present method has been adopted on shells and plates [31–36].

Sections 2–6 illustrate the fundamentals of the method in question. Adopting the three dimensional equations of motion and a series expansion assumption, a hierarchy of variationally consistent approximate equations of motion and pertinent boundary conditions are derived from Hamilton's principle. The issue whether these sets of equations are asymptotically correct or not are illustrated and discussed in Section 7. The rest of the paper concentrates mainly on the flexural motion. Section 8 presents and discusses the differential equation for the present transverse beam theory using the lowest truncation order, together with equations for other beam theories given in the literature. Numerical results are given in Section 9, covering dispersion curves, eigenfrequencies, mode shapes, motion due to prescribed end displacement and a static deflection case. For the present theory, these examples illustrate both the benchmark property of the higher order truncations, and the efficiency of the lower order engineering equations.

2. Hamilton's principle

Consider a circular cylinder with length L and radius a. The cylinder is homogeneous, isotropic and linearly elastic with density ρ and Lamé parameters λ and μ. Cylindrical coordinates are used with radial coordinate r, circumferential coordinate θ and axial coordinate z. The corresponding radial, circumferential and longitudinal displacement fields are denoted by u, v and w.

A generalized Hamilton's principle is to be used to derive variationally consistent sets of differential equations describing the motion of the cylinder together with the corresponding boundary conditions. Hereby, the governing sets of cylinder equations are treated in a unified manner. Simultaneous and independent variations of displacements and stresses are adopted such that displacement and stress boundary conditions are treated similarly [30,37,38]. The Hamilton's principle thus results in the variational expressions

$$\int_{t_0}^{t_1}\left(\int_V (\nabla\cdot\boldsymbol{\sigma}+\rho\boldsymbol{f}-\rho\ddot{\boldsymbol{u}})\cdot\delta\boldsymbol{u}\,\mathrm{d}V+\int_{S^t}\left(\hat{\boldsymbol{t}}-\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)\cdot\delta\boldsymbol{u}\,\mathrm{d}S+\int_{S^u}\left(\hat{\boldsymbol{u}}-\boldsymbol{u}\right)\cdot\delta\boldsymbol{t}\,\mathrm{d}S\right)\mathrm{d}t=0. \tag{1}$$

Here $\boldsymbol{\sigma}$ is the stress, $\boldsymbol{f}$ is the volume force and $\boldsymbol{n}$ is the unit normal vector. A prescribed traction $\hat{\boldsymbol{t}}$ acts on a subsurface denoted by S^t and a prescribed displacement $\hat{\boldsymbol{u}}$ acts on the complementary surfaces S^u. Since the virtual displacement components in $\delta\boldsymbol{u}$ and the virtual traction components $\delta\boldsymbol{t}$ are independent, Eq. (1) reduces to separate equations for each variational term. For sake of clarity, each equation is written below on component form. The equations of motion contained in the volume integrals are thus

$$\int_V\left(\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\rho f_r-\rho\frac{\partial^2u}{\partial t^2}\right)\delta u\,r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z=0, \tag{2}$$