Wave Motion 51 (2014) 256–265

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

A Higdon-like non-reflecting boundary condition for the Klein–Gordon equation with evanescent waves

John R. Dea *

Department of Mathematics and Statistics, Air Force Institute of Technology, 2950 Hobson Way, Wright-Patterson Air Force Base, OH 45433, USA

HIGHLIGHTS

- Augmented Higdon NRBC to absorb evanescent waves.
- Stability and absorption properties analyzed.
- Applied to 2-D Klein–Gordon equation in a finite difference scheme.
- Accuracy and long-time stability demonstrated by examples.

ARTICLE INFO

Article history:
Received 19 October 2012
Received in revised form 11 July 2013
Accepted 1 August 2013
Available online 14 August 2013

Keywords:
Absorbing boundary conditions
Wave propagation
Evanescent waves
Klein–Gordon equation
Open domains
Finite differences

ABSTRACT

The non-reflecting boundary condition developed by Higdon and automated by Givoli and Neta is highly effective at absorbing propagating waves in a finite difference setting, but it does not absorb evanescent waves. In this paper, we augment the Higdon scheme with additional terms to absorb these evanescent waves in the context of the two-dimensional Klein–Gordon equation. Numerical examples illustrate the performance of this technique.

1. Introduction

When modeling wave phenomena in large or unbounded domains, the phenomena of interest are frequently confined to a small portion of that domain. To minimize the required computational overhead, some non-physical method of truncating this domain becomes necessary. Research and development of such methods have been ongoing for several decades. Ideally, a domain-truncation method will be fast, accurate, stable, and easy to implement. Realistically, one must often choose two or three of those attributes at the expense of the remainder. As a result, efforts to improve each facet will continue, offering modelers a range of choices rather than a single one-size-fits-all technique.

These techniques fall broadly into one of three categories, computational tricks, a non-physical absorbing region surrounding the domain, and a non-physical boundary condition designed to mimic one-way wave propagation. One of the simplest and most intuitive computational tricks was performed by Smith in 1974 [1]. Apply a Dirichlet boundary condition, which totally reflects a propagating wave but reverses the sign of its amplitude, to one copy of the solution. Apply a Neumann boundary condition, which totally reflects a propagating wave and maintains the amplitude's sign, to another copy. Add the

* Tel.: +1 937 255 3636x4584; fax: +1 937 656 4413.
E-mail addresses: john.dea@afit.edu, john1v6@sbcglobal.net.

0165-2125/$ – see front matter. Published by Elsevier B.V.
http://dx.doi.org/10.1016/j.wavemoti.2013.08.002

two results together, and the two total reflections cancel each other, leaving a reflection-free domain. The limiting factor here is the order of accuracy of the Neumann condition's discretization.

The absorbing layer method surrounds the computational domain with a non-physical computational domain. Within this added domain, additional terms are incorporated into the computations, causing waves propagating into the regions to diminish to zero before reflecting back into the physical computational domain. The best-known method of this type is the Perfectly Matched Layer (PML), first introduced by Bérenger [2] for Maxwell's equations and later applied by other researchers to other wave-propagation problems such as the linearized shallow water equations [3] and the Euler equations [4]. Here, the limitation is the size of the absorbing layers.

A non-reflecting boundary condition (NRBC) uses a differential operator to force the system's behavior at the computational domain's non-physical boundary. This condition can involve applying a theoretically high-order differential operator to the state variable (e.g., Engquist–Majda [5], Bayliss–Turkel [6], and Higdon [7]) or by converting the high-order operator into a system of low-order operators by means of auxiliary variables (Givoli–Neta[8] and Hagstrom–Warburton [9]) or a continued-fraction iteration method (Guddati–Tassoulas [10]). In practice, high-order derivative operators are usually limited by the algebraic complexity of their implementation as well as stability issues when coupled with lower-order interior discretization schemes. For example, neither Engquist–Majda nor Bayliss–Turkel have been implemented in practice beyond second order [11], and Higdon's scheme was also limited to third order prior to Givoli and Neta's automation of the derivative discretization [12]. In each case, the accuracy is limited by the order of the imposed derivatives or the number of auxiliary variables.

All of these methods effectively absorb propagating waves. However, not all waves propagate. Consider, for example, the Klein–Gordon equation in two spatial dimensions:

$$\partial_{tt} u = C_0^2 \nabla^2 u - f^2 u \tag{1}$$

where C_0 is the principal wave speed of the system, and f represents the dispersion factor. Throughout this paper, we use the following shorthand notation for partial derivatives:

$$\partial_t = \frac{\partial}{\partial t} \qquad \partial_{xy} = \frac{\partial^2}{\partial x \partial y}. \tag{2}$$

This equation admits propagating waves, solutions which have the form $u(x, y, t) = \exp(\mathrm{i}(kx + ly - \omega t))$, where the frequency ω and wave numbers k and l satisfy the dispersion relation $\omega^2 = C_0^2(l^2 + k^2) + f^2$. The equation also admits *evanescent waves* of the form $u(x, y, t) = \exp(-kx + \mathrm{i}(ly - \omega t))$ with the dispersion relation $\omega^2 = C_0^2\left(l^2 - k^2\right) + f^2$. Since these waves may also be present in an open domain, an absorbing boundary method must account for their presence to model the infinite domain properly.

In this paper, we begin with Higdon's NRBC differential operator [7], including its subsequent developments by Givoli and Neta et al. [13,12,14,15], and augment it with a set of additional differential operator terms designed to absorb evanescent waves which can appear in solutions of the Klein–Gordon equation. After deriving the form for these differential operator terms, we will analyze some of their properties, including absorption and stability. Numerical examples will demonstrate the method's effectiveness.

The derivation and implementation of this method are similar to the evanescent mode extension to the Hagstrom–Warburton (HW) NRBC developed in [16] (see also [17,18]). In comparing this method with the HW NRBC, we have a trade-off between ease of implementation and computational overhead. Specifically, this method has the following advantages and disadvantages relative to the HW NRBC:

+ This method implements the normal derivative terms directly, avoiding the complicated algebra required to remove them in an auxiliary-variable-based scheme.
+ This method can easily handle the corner of two adjacent open boundaries, whereas auxiliary variable methods encounter severe difficulty at these corners.
× The new method is significantly more cumbersome computationally. For an NRBC of order $J + K$ (order J for propagating waves, order K for evanescent waves), the computational overhead of the new method is $O(J^3K)$; the HW NRBC method is only $O(J + K)$.
× The new method requires high-order derivatives, rendering it unusable in a finite-element system and potentially unstable in some finite-difference settings (see Section 3.3).

The rest of the paper is organized as follows. In the next section we give an overview of the existing Higdon NRBC method and point out some of its important features. We then extend it with a set of evanescent mode terms in Section 3. The finite difference implementation algorithm is presented in Section 4, followed by numerical examples in Section 5 which imply superconvergence with the given configuration of NRBC and interior discretization. Section 6 summarizes the results and proposes related topics for future research.