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On the use of perfectly matched layers in the presence of long or backward propagating guided elastic waves

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HIGHLIGHTS

- We analyze Perfectly Matched Layers in elastic waveguides and time-harmonic regime.
- The boundary conditions at the end of the layer are designed to avoid the coupling of modes.
- PMLs do not select the outgoing solution in the presence of backward propagating modes.
- PMLs can be used however to compute a kind of reduced basis of solutions of the equations.
- The outgoing solution is recovered a posteriori as a linear combination of these solutions.

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ABSTRACT

An efficient method to compute the scattering of a guided wave by a localized defect, in an elastic waveguide of infinite extent and bounded cross section, is considered. It relies on the use of perfectly matched layers (PML) to reduce the problem to a bounded portion of the guide, allowing for a classical finite element discretization. The difficulty here comes from the existence of backward propagating modes, which are not correctly handled by the PML. We propose a simple strategy, based on finite-dimensional linear algebra arguments and using the knowledge of the modes, to recover a correct approximation to the solution with a low additional cost compared to the standard PML approach. Numerical experiments are presented in the two-dimensional case involving Rayleigh–Lamb modes.

1. Introduction

Since their introduction by Bérenger [1], the perfectly matched layers (PML) have been applied to a large number of time-dependent and time-harmonic wave-equation problems set on unbounded domains. For such problems, it is necessary, if any computation is to be done, to put artificial boundaries at some distance away from the given region of interest and, in order to give accurate and reliable results, this truncation has to produce an error on the solution as small as possible. The PML provide a way to do so by introducing layers surrounding the domain of interest in which the waves enter without reflection and decay exponentially, hence solving the difficult task of choosing an adequate boundary condition at the end of the computational domain. Moreover, they are relatively easy to implement in conjunction with virtually any conventional approximation method, like the finite difference, finite element or spectral methods, and can be adapted to solve problems originating from electromagnetism, acoustics, or elasticity, to name a few. They are also generally very efficient and often compare favorably with other existing techniques for artificially handling unbounded domains (see for instance the Ref. [2],

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in which a comparison of the performances of high-order absorbing boundary conditions and several types of perfectly matched layers in two dimensions, for problems governed by the Helmholtz equation, is offered).

Despite this undeniable success, the PML technique has been shown to fail in some specific situations. For linear elastic systems in which the propagative medium presents particular anisotropy properties, numerical instabilities can be observed in time-domain simulations [3]. In the context of waveguides, anisotropy of the material is not even a necessary feature for this phenomenon to occur. Investigations connected this behavior to the existence of so-called *backward waves*, whose group and phase velocities have opposite signs.[1] In their presence, exponential growth occurs in the layers, rendering the method completely unusable [8,9]. While the generated instabilities are largely discernible in transient simulations, it should be emphasized that it is not the case in time-harmonic ones, the solution effectively computed, usually in a numerically stable manner, simply being not an approximation to the *outgoing* solution of the problem. Additionally, one should mention that the PML also perform very poorly when so-called *long waves* (associated with a mode which has an almost zero propagation constant) arise near cut-off frequencies, as the slow decay of these in the PML region calls for a very thick layer (and thus expensive computations in practice).

In the present article, we are interested in the numerical solution of time-harmonic problems set in (semi-)infinite waveguides. We introduce an original methodology based on a previous idea (see [10]), which can be seen as a way to rehabilitate the use of PML in the presence of backward waves and/or a means to improve its performance when long waves exist, at a moderate additional computational cost. It makes essential use of the orthogonality (or biorthogonality) properties enjoyed by the guided modes and the *a priori* knowledge that some of these modes are associated with backward and/or long waves. It therefore bears some strong similarities with the method proposed by Skelton et al. in [8] to overcome the very same issue, which uses the biorthogonality relations satisfied by the Rayleigh–Lamb modes to separate the forward propagating waves from the backward ones in order to treat them appropriately within the PML. It also shares a bond with the approach proposed by Barnett and Greengard in [11] for an integral representation for quasi-periodic scattering problems, in the sense that it involves the computation of a finite number of "corrections", which measure in some way the failure of the approximate solution to satisfy a radiation condition.

Our presentation will be focused on the case of an elastic waveguide, which is particularly interesting as the potential applications are numerous, notably for the detection of cracks within plates, rods, or pipes, in nondestructive testing, but the underlying idea is quite general and can be applied to other wave propagation models. Note however that, due to pending open theoretical questions on modal expansion series of guided elastic modes, we were not able to give a rigorous mathematical justification of the method, as we achieved in [10] with the same technique applied to the use of Robin-type boundary conditions as approximate radiation condition at finite distance for the Helmholtz equation. As a consequence, several essential facts need to be assumed or conjectured for the method to be applicable in the present context.

Our paper is organized as follows. In Section 2, the general setting of the problem is given and the modal formalism to be used throughout is recalled. The PML technique and its drawbacks are described in Section 3 and the novel methodology is presented in Section 4. Details on its implementation are provided in Section 5 and a few numerical results are shown in the following section. Finally, we address in the closing section some mathematical questions concerning this work which, to the best of our knowledge, remain open.

2. General setting

In this paper, the elastic wave propagation problems we aim at solving numerically can be either radiation or scattering problems, in which one wants to determine respectively the field generated by a compactly supported source placed in the waveguide or the scattered field due to a local perturbation of the waveguide given an incident field at infinity. The present section is devoted to their mathematical modeling and properties.

2.1. The elastic waveguide

For the sake of simplicity, we consider an isotropic elastic waveguide of semi-infinite length, an extension to the infinite case being dealt with in Section 5. Let $\Omega \subset \mathbb{R}^d$, with $d = 2$ or 3, be a connected unbounded domain, obtained by locally perturbing the perfectly straight waveguide, whose cross section S is a bounded subset of $\mathbb{R}^{d-1}$. More precisely, we set $\Omega_0 = \Omega \cap \{\boldsymbol{x} = (\boldsymbol{x}_S, x_d) \mid x_d < 0\}$ and $\Omega_+ = \Omega \cap \{\boldsymbol{x} = (\boldsymbol{x}_S, x_d) \mid x_d > 0\}$, Ω_0 being a bounded domain possibly containing a localized perturbation of the cylindrical geometry of the waveguide, either a deformation of the boundary or a defect (a crack for instance) enclosed in the guide (see Fig. 1 for an example).

Within the framework of linear elasticity theory, a time-harmonic dependence of pulsation $\omega > 0$ being assumed, the (vectorial) displacement is the quantity $\mathrm{Re}(\boldsymbol{u}(\boldsymbol{x})\, e^{-\mathrm{i}\omega t})$, where the field $\boldsymbol{u}$ satisfies the following equations (note that the

[1] The possibility of their existence having been first discussed by Lamb [4], it is now well-known that such waves are present in linear elasticity (see [5] for instance), as the inspection of the dispersion curves of Rayleigh–Lamb modes for some homogeneous, isotropic elastic plates shows that, for a majority of commonly encountered materials and in some frequency ranges, the phase and the energy of a given mode can propagate in opposite directions. These "backward-waves" modes also appear in dielectric-loaded circular electromagnetic waveguides, as discovered in [6], or in layered elastic structures (see [7]).