Wave Motion 51 (2014) 323–334

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Low-frequency wave propagation in post-buckled structures

Florian Paul Robert Maurin, Alessandro Spadoni *

Institute of Mechanical Engineering, École Polytechnique Fédérale de Lausanne (EPFL), 1015 Lausanne, Switzerland

HIGHLIGHTS

- Wave propagation in initially-post-buckled structures.
- Buckling pattern simultaneously provides dispersion and nonlinearity.
- Solitons.
- Solitary waves with dispersive front.

ARTICLE INFO

Article history:
Received 9 April 2013
Received in revised form 19 August 2013
Accepted 28 August 2013
Available online 11 September 2013

Keywords:
Solitons
Post-buckled structures
Dispersion
KdV

ABSTRACT

Nonlinear wave propagation in solids and material structures provides a physical basis to derive nonlinear canonical equations which govern disparate phenomena such as vortex filaments, plasma waves, and traveling loops. Nonlinear waves in solids however remain a challenging proposition since nonlinearity is often associated with irreversible processes, such as plastic deformations. Finite deformations, also a source of nonlinearity, may be reversible as for hyperelastic materials. In this work, we consider geometric bucking as a source of reversible nonlinear behavior. Namely, we investigate wave propagation in initially compressed and post-buckled structures with linear-elastic material behavior. Such structures present both intrinsic dispersion, due to buckling wavelengths, and nonlinear behavior. We find that dispersion is strongly dependent on pre-compression and we compute waves with a dispersive front or tail. In the case of post-buckled structures with large initial pre-compression, we find that wave propagation is well described by the KdV equation. We employ finite-element, difference-differential, and analytical models to support our conclusions.

1. Introduction

Nonlinear dynamic deformations of solids and material structures imply amplitude-dependent motion. Kinematic sources of nonlinearity may be accelerating local inertial frames and large deformations [1]. Kinetic sources of nonlinearity may be inelastic material behavior [2–4] or multi-field interactions such as ferroelectric and ferromagnetic effects in solid crystals [5,6], and phase transitions in martensitic [7] or shape-memory alloys [8,9]. All these forms of nonlinearity may produce non-stationary processes and chaotic motion [10], but stable, nonlinear waves may exist if nonlinearity sources can be balanced by attenuation such as dissipation or dispersion. Typical stable solutions include nonlinear periodic waves, solitons, and bell-shaped solitary waves [1]. Solitons and bell-shaped solitary waves are solutions that have attracted significant interest since they preserve their amplitude as they travel with amplitude-dependent velocity, and they are unchanged by collisions with other such waves [11].

In the case of continuous solids and structures, the propagation of nonlinear waves requires nonlinear-interaction potentials both for kinetic and kinematic sources of nonlinearity in a non-accelerating inertial frame. Inelastic behavior

* Corresponding author. Tel.: +41 21 693 38 56.
E-mail address: alex.spadoni@epfl.ch (A. Spadoni).

0165-2125/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.wavemoti.2013.08.008

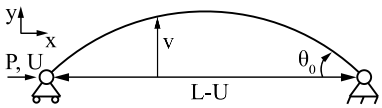

Fig. 1. Axially-loaded slender beam with large-deformation solution resulting from the elastica [26].

often results in the propagation of shock waves [12], and this is an irreversible process. For example, surface waves steep shock fronts can generate cracks that are used to measure the fracture strength of anisotropic crystals [13,14]. Geometric nonlinearities and irreversible material behavior are activated by large loads on certain slender continuous structures leading to the propagation of bucklewaves [15]. Reversible nonlinear-wave processes, on the other hand, have been reported in either nonlinear solid crystals [16] or in waveguides like rods and plates made of nonlinear-elastic materials like Mooney or Murnaghan materials [11]. In the first case, dispersion induced by interatomic spacing – a natural length scale – and nonlinearity arising from anharmonic atomic potentials lead to solitons. In nonlinear rods and plates, solitons result from dispersion introduced by geometric properties of the waveguides – the rod radius and the plate thickness – and nonlinear material behavior. Solitons have also been demonstrated in thin-film-like coatings resting on a nonlinear substrate made of lithium niobate [17]. Stable nonlinear waves are also predicted as bulk waves in general hyperelastic, power-law solids [18]: in this case nonlinearity is induced by material behavior and dispersion arises from coupling of deformation components. The coupling of deformation with pre-stress is also an important phenomenon for waves of infinitesimal amplitude in hyperelastic [19] and incompressible materials [20].

Slender structures such as rods also allow reversible nonlinear processes such as propagating curves or loops with the same characteristics as solitons [21–24]. Nonlinearity comes from large deformations, while strain remains small, and dispersion is provided by curvature resulting from the traveling loop. While these solutions have yet found no application, different types of solitary waves are found depending on the rod formulation. In [23], the modified Korteweg–de Vries (mKdV) equation is found which is a modified version of the prototypical equation (KdV) describing stable waves resulting from the interaction of nonlinearity and dispersion [11]. This is a relevant problem as nonlinear waves in solids provide a testbed to explore increasingly complex nonlinear models such the modified KdV equation which describes numerous phenomena like water waves – for which the KdV was first derived – and vortex filaments for example [25]. In [24], the so-called $K(m, n)$-KdV equation is found admitting solitary waves with compact support called compactons.

We introduce here an additional state of material structures which has the necessary characteristics to host stable, nonlinear, elastic waves: the post-buckled state. Buckled slender structures have intrinsic length scales dependent on the buckling pattern and this naturally introduces simultaneously dispersion and nonlinearity. Buckled slender structures indeed have a nonlinear load–deformation relation resulting from geometric nonlinearities. For rods, buckling critical load and load–deformation relationships are given in [26] employing Bernoulli or Timoshenko beam theory. Other modes of geometric instability such as barreling are to be obtained for stubby structures via the Stroh formalism [27]. In this paper we investigate wave solutions in the long-wavelength regime with analytical models, which we show to coincide with the KdV, and solutions of time-integrated finite-element (FE) models. We consider the constitutive properties to be linear elastic.

This article is organized as follows: in Section 2, static solutions for buckled beams are summarized and FE models are described in Section 3. In Section 4, a homogenized model for wave propagation in buckled beams is derived. Finally, results are discussed in Section 5 where FE predictions (which we consider exact for our purposes) are compared to both discretized and homogenized analytical models. We find that both approximate models agree with FE results only for large initial deformations. For large pre-stress, we find solutions as propagating waves which correspond to the soliton solution of the KdV equation. When the initial compression is moderated or small, a wave packet always precedes the main waveform and agreement between FE and approximate models is lost. This may result from certain sources of dispersion that are not included in the KdV equation.

2. Post-buckled configurations

In this work, we limit ourselves to investigating one-dimensional (1D) post-buckled structures with linear-elastic material behavior for simplicity. Consider the simply-supported, slender, Bernoulli beam shown in Fig. 1, with cross-sectional area A, Young's modulus E, area-moment of inertia I, density ρ, and initial length L. An external axial load P at the extremity produces an axial displacement U. Considering infinitesimally small, static deformations and accounting for coupling of axial loads and transverse deformations $v(x)$, one obtains [26]

$$EI\frac{d^4 v(x)}{dx^4} + P \cdot \frac{d^2 v(x)}{dx^2} = 0, \tag{1}$$

with solution

$$v(x) = C_1 \cos \gamma x + C_2 \sin \gamma x + C_3 x + C_4, \tag{2}$$