Wave Motion 51 (2014) 348–359

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Complex wavenumber Fourier analysis of the B-spline based finite element method

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HIGHLIGHTS

- Dispersion analysis of B-spline FEM in 1D elastic wave propagation is presented.
- Influence of the number of control points, their positions and C^0 connection is studied.
- Optimizing procedure for obtaining minimal dispersion errors is suggested.
- Found that B-spline parameterizations produce a superior dispersion behaviour.

ARTICLE INFO

Article history:
Received 15 April 2013
Received in revised form 27 August 2013
Accepted 13 September 2013
Available online 3 October 2013

Keywords:
Elastic wave propagation
Dispersion errors
B-spline
Finite element method
Isogeometric analysis

ABSTRACT

We present the results of one-dimensional complex wavenumber Fourier analysis of the B-spline variant of Finite Element Method (FEM). Generally, numerical results of elastic wave propagation in solids obtained by FEM are polluted by dispersion and attenuation. It was shown for the higher-order B-spline based FEM, that the optical modes did not occur in the case of infinite domains, unlike the higher-order Lagrangian and Hermitian finite elements, and also the dispersion errors are smaller. The paper's main focus is on the wave propagation through B-spline multi-patch/segment discretization with the C^0 connection of B-spline segments and, chiefly, to the determining of dispersion and attenuation dependences. The numerical approach employed leads to substantial minimization of dispersion errors. Furthermore, the errors decrease in line with the increasing order of the B-spline elements/segments, with the local refinement, and also by the particular choice of the positions of control points through the optimizing procedure.

1. Introduction

The paper is concerned with the numerical dispersion phenomena of the Finite Element Method (FEM) based on the continuous Galerkin formulation [1] and its modification, where B-spline basis functions such as shape functions are utilized [2]. When the propagating phenomena in solids are modelled by FEM, the speed of a monochromatic wave depends on its frequency as well as on wavenumber [3]. This phenomenon is called dispersion. In principle, the dispersion errors in numerical solutions of wave propagation are caused by both spatial and temporal discretizations [1,4]. The sequel of FEM dispersion properties is that a propagated wave packet is distorted. Furthermore, the finite element (FE) mesh behaves as a frequency filter — higher frequencies are transferred with a strong attenuation [4]. On the other hand, a monochromatic harmonic stress wave propagates in an unbounded elastic continuum regardless of its frequency and wavenumber and, correspondingly, a wave packet propagates without distortion. The 'ideal' 1D continuum is dispersionless — meaning that the wave speed is constant, independent of frequency as well as of wavenumber [5].

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0165-2125/$ – see front matter
http://dx.doi.org/10.1016/j.wavemoti.2013.09.003

Dispersion properties of 1D constant strain (linear) FEs employed in the numerical solution to the scalar wave equation have been studied by Krieg and Key [6]. Furthermore, Belytschko and Mullen have analysed the 1D dispersion analysis of higher-order (quadratic) FEs [3]. The existence of optical branches in the frequency spectrum has been proved. This notion refers to Brillouin's original book [7] in which the lowest characteristics were called *acoustic* branches and the higher ones *optical* branches. In that study, the dispersion analysis revealed the stop band and band gap in the frequency spectrum of biquadratic elements, whose corresponding solutions in the frequency range decayed exponentially. Generally in FEM discretization, the pass filter behaviour is a pure numerical artefact. The physical meaning of higher dispersion branches and their basic behaviour have been studied by Abboud and Pinsky [8]. Moreover, the magnitude elongation effect of quadratic finite elements has been observed there. This effect does not exist for the linear ones. The dispersion analysis of higher order (p-version) finite elements has been realized by Thompson and Pinsky [9]. One-dimensional Lagrangian and Hermitian FEs have been studied even before by Okrouhlík and Höschl [10]. Furthermore, the behaviour of classical higher-order FEs accounts for discontinuities in their spectra as well as for false representation of maximum frequency, whose error increases with element order. For brevity this property is referred to as divergent behaviour in the text. In multidimensional cases, we have to take into account the anisotropy dispersion behaviour of FEM [11].

A number of the FEM modifications for the numerical solution of wave-like equations are known. In seismology, spectral finite element method (SEM) appeared recently [12]. SEM is of h-type Lagrangian FEM, where nodes have special positions along the FEs corresponding to the numerical quadrature schemes. But the displacements along elements are approximated by the Lagrangian interpolation polynomials. If the set of nodes takes positions with respect to the Gauss–Lobatto–Legendre quadrature points, the mass matrix has a diagonal form. SEM improves the dispersion errors for lower branches, but not for the upper ones [13]. In SEM, shape functions in the form of the Legendre polynomials and the hierarchic Fourier functions have been tested by Thompson and Pinsky [9].

A modern approach to computational continuum mechanics is the Isogeometric Analysis (IGA) [2], taking its inspiration from Computer Aided Design (CAD). This modification of FEM employs shape functions of splines (for examples, B-splines, NURBS-Non-Uniform Rational B-spline, T-splines and others). The advantage of spline-oriented FEM is that geometry and approximation of the field of unknown quantities are prescribed by the same technique. Another benefit is that the approximation is smooth with a higher order of continuity and also some 2D and 3D geometries could be precisely described [14]. IGA aims at integrating FEM ideas in CAD systems without the necessity of generating new computational meshes as required by the conventional FEM.

It has been shown that IGA produces optimal convergence rate and dispersion properties in elastodynamics problems [15–17]. B-spline based FEM exhibits small dispersion errors in wave propagation in a one-dimensional unbounded elastic medium [15,18,19]. It has also been shown that the higher *optical* modes did not exist and, subsequently, dispersion errors were reported to decrease with the increasing order of B-spline shape functions [15–17]. IGA improves the dispersion errors and frequency spectrum in comparison to higher-order Lagrangian finite elements. This can be ascertained from the fact that B-spline basis functions are uniform (homogeneous) for a uniform knot vector. This means that constituent basis functions are the same, but they are moved in the parametric space with an identical distance [14]. For this reason, equations of motion are repeated and the discretized medium subsequently produces a low level of dispersion errors with no stopping bands [15]. On the other hand, the high order of continuity of B-spline basis functions generates not only propagating waves, but also evanescent waves with a pure imaginary wavenumber. With respect to the high level of attenuation of evanescent waves, the IGA numerical solutions are not being polluted by these waves [15]. Generally, the B-spline or NURBS basis functions for bounded domains are not uniform [14]. For this reason, the non-homogeneity of basis functions close to the boundary of the domain produces the dispersion and attenuation behaviour.

Based on an extensive study of the literature, we have opted to contribute to the subject with a complex wavenumber Fourier analysis of the B-spline based FEM in elastic wave propagation through a 1D infinite domain divided into segments with the C^0 connection. The main attention of the dispersion analysis is devoted to the study of influence of the C^0 continuity between spline segments and also of parameterization inside the B-spline segments. The results of the 1D dispersion analysis could serve to set multi-patch B-spline presentations in 2D and 3D numerical modelling of elastic wave propagation problems and also for the accurate modelling of non-linear wave propagation and solitary waves. In Section 2, an overview of the B-spline representation is inspected. The fundamentals of 1D elastic wave propagation and the B-spline based FEM in the 1D case are presented in Section 3. The theoretical background to dispersion analysis is mentioned in Section 4. The results of the complex wavenumber Fourier analysis of the B-spline based FEM are presented in Section 5. The paper closes with conclusions in Section 6.

2. B-spline representation

A B-spline curve is a piecewise polynomial curve with the higher-order continuity between the individual elements/segments that are differentiable up to a prescribed order [14]. Point coordinates of a B-spline curve $\mathbf{C} \in \mathbb{R}^d$ are given by the linear combination of B-spline basis functions $N_{i,p}$, so that

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\,\mathbf{P}_i, \tag{1}$$