Wave Motion 51 (2014) 1–13

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

On the efficient representation of the half-space impedance Green's function for the Helmholtz equation

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HIGHLIGHTS

- A new form of the impedance Green's function for the Helmholtz equation is presented.
- The new hybrid formula combines images in physical space with a Sommerfeld integral.
- A fast algorithm for the numerical evaluation of the Green's function is outlined.
- Numerical examples are performed to demonstrate the accuracy of our representation.

ARTICLE INFO

Article history:
Received 29 November 2012
Received in revised form 25 March 2013
Accepted 29 April 2013
Available online 15 May 2013

Keywords:
Helmholtz
Impedance
Green's function
Layered media
Sommerfeld integral
Half-space

ABSTRACT

A classical problem in acoustic (and electromagnetic) scattering concerns the evaluation of the Green's function for the Helmholtz equation subject to impedance boundary conditions on a half-space. The two principal approaches used for representing this Green's function are the Sommerfeld integral and the (closely related) method of complex images. The former is extremely efficient when the source is at some distance from the half-space boundary, but involves an unwieldy range of integration as the source gets closer and closer. Complex image-based methods, on the other hand, can be quite efficient when the source is close to the boundary, but they do not easily permit the use of the superposition principle since the selection of complex image locations depends on both the source and the target. We have developed a new, hybrid representation which uses a finite number of real images (dependent only on the source location) coupled with a rapidly converging Sommerfeld-like integral. While our method applies in both two and three dimensions, we restrict the detailed analysis and numerical experiments here to the two-dimensional case.

1. Introduction

A number of problems in acoustics (and electromagnetics) involve the solution of the Helmholtz equation,

$$(\Delta + k^2)u(\boldsymbol{x}) = f(\boldsymbol{x}), \tag{1.1}$$

in the half-space $P = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ or $S = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, subject to suitable boundary and radiation conditions. In acoustics, the Helmholtz coefficient k is given by $k = \frac{\omega}{c}$, where ω is the governing angular frequency (assuming a time-harmonic motion dependency of $e^{-i\omega t}$) and c is the sound speed. In the present paper, we assume $k \in \mathbb{C}$ is constant throughout the region of interest, with $\mathrm{Re}(k) \geq 0$ and $\mathrm{Im}(k) \geq 0$. For concreteness, we concentrate initially on the two-dimensional problem of computing the scattered field due to a unit-strength point source located at $\boldsymbol{x}_0 = (x_0, y_0)$ in the presence of a "sound-hard" obstacle over an infinite half-space subject to impedance boundary conditions (Fig. 1).

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0165-2125/$ – see front matter
http://dx.doi.org/10.1016/j.wavemoti.2013.04.012

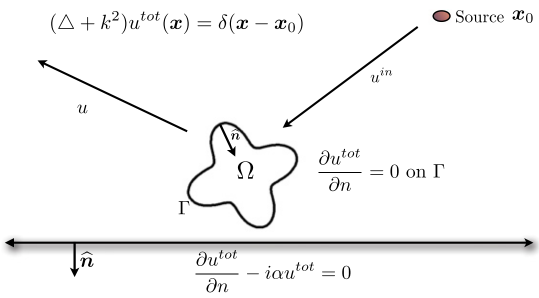

Fig. 1. Scattering from a *sound-hard* obstacle above an impedance plane.

We let the total field be defined as $u^{tot} = u^{in} + u$, where u^{in} denotes the (known) incoming field due to the point source and u denotes the scattered field. On a sound-hard obstacle Ω with boundary Γ, the total field must satisfy homogeneous Neumann boundary conditions. Since the scattered field involves no sources outside Ω, it must satisfy the homogeneous Helmholtz equation

$$(\Delta + k^2)u(\boldsymbol{x}) = 0 \tag{1.2}$$

for $\boldsymbol{x} \in P \setminus \Omega$. On the obstacle boundary Γ, we have

$$\frac{\partial u}{\partial n} = -\frac{\partial u^{in}}{\partial n}, \tag{1.3}$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative. Finally, on the interface, we assume a standard impedance condition on the total field of the form:

$$\frac{\partial u^{tot}}{\partial n} - i\alpha u^{tot} = 0. \tag{1.4}$$

Since the interface is the x-axis, we have $\frac{\partial}{\partial n} = -\frac{\partial}{\partial y}$. In physically-motivated problems, an impedance condition is typically used to approximate a more complicated wave/surface interaction, such as scattering from a rough surface, an underlying porous medium, a complicated surface coating, etc. (see [1,2]). In many applications, $\alpha = \beta k$, with $0 \leq \beta \leq 1$. The parameter β in this context is called the surface admittance. In general, depending on the physical model, β can be real or complex. For the purposes of this paper, we will assume that $\alpha \in \mathbb{C}$, with $\mathrm{Re}(\alpha) \geq 0$, $\mathrm{Im}(\alpha) \geq 0$, $|\alpha| \leq |k|$, and leave aside any further discussion of the modeling. The Green's function analysis of the present paper can be generalized to other values of α, but we restrict our attention to α in the indicated range for the sake of simplicity. A second simplification is that we only consider the case of constant α (i.e. we do not permit α to vary along the length of the half-space interface). There is a substantial literature on impedance problems and we mention only a few relevant papers which also discuss the computation of the corresponding Green's function. These include [3–11].

Returning now to the scattering problem (1.2)–(1.4), an ansatz for the solution is to represent the total field as

$$u^{tot}(\boldsymbol{x}) = \int_\Gamma g_{k,\alpha}(\boldsymbol{x}, \boldsymbol{y})\, \sigma(\boldsymbol{y})\, ds_{\boldsymbol{y}} + u^{in}(\boldsymbol{x}), \tag{1.5}$$

where s is arc length along Γ, $g_{k,\alpha}(\boldsymbol{x}, \boldsymbol{x}_0)$ is the Green's function for the half-space P with homogeneous impedance boundary conditions, and $u^{in}(\boldsymbol{x}) = g_{k,\alpha}(\boldsymbol{x}, \boldsymbol{x}_0)$. Imposing the Neumann conditions (1.3) on Γ yields the Fredholm integral equation of the second kind:

$$-\frac{1}{2}\sigma(\boldsymbol{x}) + \int_\Gamma \frac{\partial}{\partial n_x} g_{k,\alpha}(\boldsymbol{x}, \boldsymbol{y})\, \sigma(\boldsymbol{y})\, ds_{\boldsymbol{y}} = -\frac{\partial}{\partial n_x} g_{k,\alpha}(\boldsymbol{x}, \boldsymbol{x}_0) \tag{1.6}$$

for $\boldsymbol{x} \in \Gamma$. Eq. (1.6) is invertible except for a countable sequence of spurious resonances $\{k_j\}$. Resonance-free, but more complicated representations are well-known [5], which we will not review here, since we are primarily interested in the question of how to efficiently evaluate the impedance Green's function $g_{k,\alpha}$ itself. In our examples, we will always assume $k \notin \{k_j\}$ and that Eq. (1.6) is solvable. Note that, by using the impedance Green's function in the integral representation, the infinite half-space boundary does *not* need to be discretized.

Algorithms for the computation of $g_{k,\alpha}$ date back to the classical work of Sommerfeld, Weyl, and Van der Pol [12,13,11], who developed both what are now referred to as the Sommerfeld integral and the method of complex images. For more recent treatments of this problem, see [14–16,8,10,17].

The main contribution of the present work is the observation that a finite number of *real* images can accurately capture the high-frequency components of the Sommerfeld integral. This leads, naturally, to a hybrid representation of the Green's function in terms of a rapidly converging Sommerfeld-type representation, augmented with $\mathcal{O}(\log(1/d))$ real images for each source point that lies a distance d from the impedance interface. Our approach is somewhat related to that of Cai and