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Operator expansions and constrained quadratic optimization for interface reconstruction: Impenetrable periodic acoustic media

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HIGHLIGHTS

- A numerical scheme for interface reconstruction of grating shapes in a two-dimensional acoustic medium is proposed.
- The algorithms are stabilized by casting the problem as a constrained quadratic optimization problem.
- Numerical simulations demonstrate the enhanced stability and accuracy of the new approach even in the presence of noise.

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ABSTRACT

Grating scattering is a fundamental model in remote sensing, electromagnetics, ocean acoustics, nondestructive testing, and image reconstruction. In this work, we examine the problem of detecting the geometric properties of gratings in a two-dimensional acoustic medium where the fields are governed by the Helmholtz equation. Building upon our previous Boundary Perturbation approach (implemented with the Operator Expansions formalism) we derive a new approach which augments this with a new "smoothing" mechanism. With numerical simulations we demonstrate the enhanced stability and accuracy of our new approach which further suggests not only a rigorous proof of convergence, but also a path to generalizing the algorithm to multiple layers, three dimensions, and the full equations of linear elasticity and Maxwell's equations.

1. Introduction

Grating scattering is a fundamental model in remote sensing [1], electromagnetics [2], ocean acoustics [3], nondestructive testing [4], and image reconstruction [5]. In a recent paper [6] we devised a Boundary Perturbation technique (based upon the Operator Expansions – OE –[7,8] formalism) for approximating solutions of two related problems in these fields: (1) the "forward problem" of simulating scattering returns of known incident radiation interacting with a grating of *known* structure; and (2) the "inverse problem" of recovering the grating structure given data about both the incident and scattered radiation. In this work we augment and, as we shall see, vastly improve our approach to (2) with the addition of a "smoothing" mechanism similar in spirit (though not identical to) classical Tikhonov regularization [9].

As with the OE method as it was originally designed by Milder [7,10–12] and Milder and Sharp [13,14], our new approach is spectrally accurate (i.e., has convergence rates faster than any polynomial order) due to both the analyticity of the scattered

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fields with respect to boundary perturbation, and the optimal choice of spatial basis functions which arise naturally in the methodology. Our inversion strategy was originally inspired by the work of Nicholls and Taber [15,16] on the recovery of topography shape under a layer of an ideal fluid (e.g. the ocean) which also uses the *explicit* nature of the OE formulas to great effect.

This contribution is most similar in spirit to the work of Ito and Reitich [17] who considered the same problem in a similar, but not identical, framework. They employed the Field Expansions [18–20] method as the underlying "forward solver", and used a regularization to realize greater stability. By contrast, we use the Operator Expansions [7,8] formalism which not only has different numerical properties (see, e.g., [21,22]), but also features formulas where the interface deformation appears *explicitly* which, as we demonstrate in Sections 4.2 *and* 4.4, lead to elegant and easily-derived inversion formulas. In addition, our regularization features a slightly different cost functional to be minimized which leads to a constrained *quadratic* optimization problem rather than a more general nonlinear one. While this comes at the cost of greater demands placed upon the measurement of the far-field data, it is rewarded with a vastly simplified numerical procedure (the "Null Space Method" of Quadratic Programming [23]) which involves inversion of easily precomputed matrices without line searches, derivative calculations, conjugate gradient implementations, nonconvexities, etc.

Other approaches to this problem have, of course, been considered and we refer to Ito and Reitich's [17] paper and the classical texts of Colton and Kress [24,25,9] for extensive lists. We mention that most of these are based upon Integral Equation (IE) formulations requiring explicit knowledge of the Green's function. These methods, like our Boundary Perturbation approach, are accurate and efficient as they posit *surface* unknowns and are *spectrally* accurate. However, we mention that our current approach is clearly advantaged compared to IE methods in two aspects:

1. for the periodic problems we consider here IE approaches *must* faithfully compute the *periodized* Greens function (e.g., via Ewald summation) which is not only technically challenging, but also introduces an additional discretization parameter thereby significantly *increasing* the computational cost. (See the extensive discussion in [26] for a full discussion of these matters together with suggestions to minimize these issues.) Due to the Fourier basis functions utilized by our scheme, these quasiperiodic solutions are computed "by default" at no additional cost.
2. as we mentioned above, the impetus for using the Operator Expansions formalism is due to the fact that the boundary deformations and their powers appear *explicitly* in this formulation. As with the Field Expansions recursions, the boundary shape appears in the IE formulation in a rather implicit fashion rendering the identification of formulas akin to those in Section 4 impossible.

Specific to the simulation of the scattered field from a grating using IE methods, most recently we are aware of the work of Arens and Grinberg [27] on the "Complete Factorization Method" as applied to periodic gratings, the iterative regularization method of Hettlich [28], and the papers of Bruckner et al. [29–31] on the generalization of the Kirsch–Kress optimization method to gratings. In this latter work, [29,31] focused upon Integral Equation formulations, while [30] modified these approaches to the Finite Element framework. We also point out the work of Kress and Tran [32], Akduman, Kress, and Yapar [33], Lahcene and Gaitan [34], and the bibliographies of these.

The organization of the paper is as follows: in Section 2 we recall the governing equations, including a discussion of relevant unknowns (Section 2.1). In Section 3 we discuss the forward problem with relevant expansions presented in Section 3.1 followed by brief numerical results in Section 3.2. In Section 4 we discuss the particulars of the inverse problem and review our previous Boundary Perturbation approach to this problem, specifically we give a formula for the mean depth in Section 4.1, a linear approximation for the perturbation shape in Sections 4.2 and 4.3, and a nonlinearly corrected version in Section 4.4. We describe our new, regularized version of this algorithm in Section 5, with the principal ideas for a linear algorithm in Section 5.1, precise details in Section 5.3 (inspired by the procedures for Constrained Quadratic Programming which we recall in Section 5.2), and a new regularized nonlinear correction given in Section 5.4. We close with extensive numerical results in Section 6.

2. Governing equations

In this contribution we focus on the problem of recovering the interface of a two-dimensional acoustic medium overlying an impenetrable, periodic solid. Without loss of generality we assume that observations may be made at $y = 0$, the interface is "centered" at the (unknown) level $y = \bar{g} < 0$, and has (unknown) deviation $g(x)$ such that

$$\bar{g} + g(x) < 0,$$

see Fig. 1.

For simplicity we focus upon the case of a "sound soft" lower material, isonified from above by time-harmonic incident radiation

$$u^i(x, y) = e^{i\alpha x - i\beta y}.$$

It is well known that the scattered field, $u = u(x, y)$, is governed by the Dirichlet problem [2,6]

$$\Delta u + k^2 u = 0 \quad y > \bar{g} + g(x) \tag{2.1a}$$

$$\partial_y u - (i\beta_D)u = 0 \quad y = 0 \tag{2.1b}$$