Wave Motion 51 (2014) 86–99

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Efficient computation of steady solitary gravity waves

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HIGHLIGHTS

- An efficient and highly accurate method for gravity solitary wave computation is proposed.
- This method allows one to reconstruct various fields under the solitary wave up to very high amplitudes.
- The behaviour of several integral quantities is studied for a wide range of wave amplitudes.

ARTICLE INFO

Article history:
Received 16 February 2013
Received in revised form 23 May 2013
Accepted 20 June 2013
Available online 28 June 2013

Keywords:
Surface waves
Gravity waves
Euler equations
Solitary wave
Petviashvili method

ABSTRACT

An efficient numerical method to compute solitary wave solutions to the free surface Euler equations is reported. It is based on the conformal mapping technique combined with an efficient Fourier pseudo-spectral method. The resulting nonlinear equation is solved via the Petviashvili iterative scheme. The computational results are compared to some existing approaches, such as Tanaka's method and Fenton's high-order asymptotic expansion. Several important integral quantities are computed for a large range of amplitudes. The integral representation of the velocity and acceleration fields in the bulk of the fluid is also provided.

1. Introduction

Solitary waves play a central role in nonlinear sciences [1]. They appear in various fields, ranging from plasmas physics [2] to hydrodynamics [3] and nonlinear optics [4]. For integrable models, it can be rigorously shown that any smooth and localised initial condition will split into a finite number of solitons plus a radiation [5]. Solitons are special solitary waves interacting elastically, i.e., subject only to phase shifts after collisions [6,5]. However, in full Euler equations, the interaction is known to be inelastic [7]. In some sense, solitons are elementary structures which span the system dynamics [8] along with (relative) equilibria [9], periodic orbits [10], etc. This is one of the main reasons why these solutions attract so much attention.

In some special cases, the solitary waves can be found analytically. For example, explicit expressions are known for integrable models such as KdV and NLS equations [11–13], but also for some non-integrable Boussinesq-type [14–16] and Serre–Green–Naghdi [17–20] equations. The examples of such analytical solutions are numerous [21]. However, no closed-form solutions are known for the practically very important case of the free surface Euler equations. Craig and Sternberg showed that solitary wave solutions to the Euler equations are necessarily positive and symmetric [22] (without surface tension effects). In order to construct these solutions, one has to apply some approximate methods. Historically, high-order

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0165-2125/$ – see front matter
http://dx.doi.org/10.1016/j.wavemoti.2013.06.007

asymptotic approximations were proposed first [23,24]. However, these solutions are asymptotic by construction and are therefore valid only in the limit $a/d \to 0$ (a being the wave amplitude, d the uniform undisturbed water depth); moreover, these series are known to be divergent [25]. In order to avoid this limitation, several numerical approaches have been proposed [26,27], such as the Dirichlet-to-Neumann operator method [28] or the boundary integral equation method [29]. High-amplitude solitary waves up to the limiting wave were studied by Longuet-Higgins and Tanaka [30], among others. One of the most widely used methods nowadays is the Tanaka algorithm [31]. In the present study, we are going to compare extensively our computational results to Tanaka's method.

The approach we proposed in a short recent preliminary study [32] is also based on the conformal mapping technique, as the Tanaka method [31], for example. However, traditionally the conformal map is coupled with the Newton method [33] to find the solitary wave profile [34–36]. Newton-type iterations require the computation of a Jacobian matrix and the resolution of linear systems of equations (by direct or iterative methods) [37]. From a computational point of view, simple iterative schemes are much easier to implement, and they require only the evaluation of operators involved in the equation to be solved. In the previous study [32], we adopted the classical Petviashvili iteration [38], in which the convergence is ensured by computing the so-called stabilising factor [39]. This iterative scheme has already been applied to compute special solutions to many nonlinear wave equations [40–43]. An interesting comparison among different methods was recently performed for the solitary waves of the Benjamin equation [44]. The combination of two main ingredients, i.e., the Petviashvili scheme together with the conformal mapping technique, allowed us to propose a very efficient numerical scheme for the computation of solitary gravity waves of the full Euler equations in water of finite depth [32]. The proposed algorithm admits a very compact and elegant implementation in Matlab, for example. The resulting script is ready to use and it can be freely downloaded from the *Matlab Central* server [45].

In the present study, we perform further tests and validations of the new algorithm. Moreover, several important integral characteristics such as the mass, momentum, and energy, are derived in the conformal space and computed numerically to the high accuracy for a wide range of solitary waves. Our method allows us also to compute efficiently important physical fields in any point inside the bulk of the fluid layer. In this way, the pressure, velocities, and accelerations are shown under a large-amplitude solitary wave, up to an arbitrarily high accuracy.

This study is organised as follows. In Section 2, we present the governing equations along with the conformal map technique. Several important integral quantities expressed in the transformed space are provided in Section 2.2. The Babenko integral equation is derived in Section 2.3. Section 3 contains a description of the numerical scheme along with some validations and tests. Finally, some conclusions of this study are outlined in Section 4.

2. Mathematical model

We consider steady two-dimensional potential flows due to surface gravity solitary waves in constant depth. The fluid is homogeneous, the pressure is zero at the impermeable free surface, and the seabed is fixed, horizontal, and impermeable.

Let (x, y) be a Cartesian coordinate system moving with the wave, x being the horizontal coordinate and y the upward vertical one. Since solitary waves are localised in space, the surface elevation tends to zero, along with all derivatives, as $x \to \pm\infty$, and $x = 0$ is the abscissa of the crest. The equations of the bottom, of the free surface, and of the mean water level are given correspondingly by $y = -d$, $y = \eta(x)$, and $y = 0$. The parameter $a \equiv \eta(0)$ denotes the wave amplitude. Since gravity solitary waves of the Euler equations are known to be symmetric and positive [22], we have $\eta(-x) = \eta(x) \geqslant 0$ and $a = \max(\eta)$.

Let ϕ, ψ, u, and v be the velocity potential, the stream function, and the horizontal and vertical velocities, respectively, such that $u = \phi_x = \psi_y$ and $v = \phi_y = -\psi_x$. It is convenient to introduce the complex potential $f \equiv \phi + \mathrm{i}\psi$ (with $\mathrm{i}^2 = -1$) and the complex velocity $w \equiv u - \mathrm{i}v$ that are holomorphic functions of $z \equiv x + \mathrm{i}y$ (i.e., $w = \mathrm{d}f/\mathrm{d}z$). The complex conjugate is denoted with a star (e.g., $z^* = x - \mathrm{i}y$), while subscripts 'b' denote the quantities written at the seabed – e.g., $z_{\mathrm{b}}(x) = x - \mathrm{i}d$, $\phi_{\mathrm{b}}(x) = \phi(x, y = -d)$ – and subscripts 's' denote the quantities written at the free surface – e.g., $z_{\mathrm{s}}(x) = x + \mathrm{i}\eta(x)$, $\phi_{\mathrm{s}}(x) = \phi(x, y = \eta(x))$. Note that, for example, $u_{\mathrm{s}} = (\partial_x\phi)_{\mathrm{s}} \neq \partial_x(\phi_{\mathrm{s}}) = u_{\mathrm{s}} + \eta_x v_{\mathrm{s}}$. We also emphasise that ψ_{s} and ψ_{b} are constants, because the surface and the bottom are streamlines.

The far-field velocity is such that $(u, v) \to (-c, 0)$ as $x \to \pm\infty$, so c is the wave phase velocity observed in the frame of reference where the fluid is at rest at infinity ($c > 0$ if the wave travels to the increasing x-direction). Note that $c = (\psi_{\mathrm{b}} - \psi_{\mathrm{s}})/d$ due to mass conservation.

The dynamic condition can be expressed in form of the Bernoulli equation

$$2\,p + 2\,g\,y + u^2 + v^2 = c^2, \tag{2.1}$$

where p is the pressure divided by the constant density ρ and $g > 0$ is the acceleration due to gravity. At the free surface, the pressure equals that of the atmosphere, which is constant and set to zero without loss of generality, i.e., $p_{\mathrm{s}} = 0$.

2.1. Conformal mapping

We adopt the change of independent variable $z \mapsto \zeta \equiv (\mathrm{i}\psi_{\mathrm{s}} - f)/c$, that conformally maps the fluid domain $\{-\infty \leqslant x \leqslant \infty; -d \leqslant y \leqslant \eta\}$ into the strip $\{-\infty \leqslant \alpha \leqslant \infty; -d \leqslant \beta \leqslant 0\}$, where $\alpha \equiv \mathrm{Re}\,(\zeta)$ and $\beta \equiv \mathrm{Im}(\zeta)$ (see