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The complete infinite series solution of systems governed by the wave equation with boundary damping

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HIGHLIGHTS

- A complete modal solution of structures governed by the wave equation with boundary damping is derived.
- Explicit expressions for the series coefficients are provided.
- A new orthogonality condition of the mode shapes is developed.
- The solution is also presented in a traveling wave form, which is the extension of the classical D'Alembert formula.

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ABSTRACT

A common method of solving initial boundary value problems is separation of variables, denoted as modal analysis in the field of flexible structures. For systems with undamped boundary conditions the method is well-established, but for systems with boundary damping it does not provide closed form solutions. In this paper the exact modal series solution for second order systems with damped boundaries is derived with explicit expressions for the series coefficients. Knowledge of these coefficients enables practical applications of the solution, such as finite dimension approximation. The key element of the derivation is a new orthogonality condition for the damped eigenfunctions. The modal series is also transformed into a traveling wave form. The solution, which is the extension of the classical D'Alembert formula, is represented by a single equivalent propagating wave. A component of the solution, denoted by "end waves", is identified to provide the continuity of the systems displacement response.

1. Introduction

The free response of systems governed by the second order linear wave equation is discussed in any classical vibration literature, e.g. [1–4]. Examples of mechanical structures governed by this equation are taught strings in transverse vibration and rods in axial or torsional vibration. The dominating approach is modal analysis that provides complete solutions for systems with conservative boundaries, such as fixed, free, or connected to a spring or a mass. The solutions are given in terms of infinite series that may be regarded as standing waves since they are a sum of oscillating mode shapes (the eigenfunctions). The series coefficients, which are the excitation levels of each mode, are obtained by the orthogonality of the eigenfunctions. For systems with non-conservative boundary conditions (BC), such as viscous damping, the application of modal analysis is less obvious and, consequently, is rarely considered in the vibration literature.

Several important advances in the field have been reported over the years. In [5], the eigenvalues, as well as the general structure of the eigenfunctions of a structure with one end damped and the other fixed were identified. The notion of

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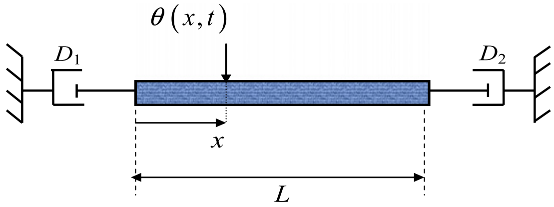

Fig. 1. A schematic diagram of a damped–damped rod.

reflection coefficients, which indicate the form of waves reflected from the boundaries, was also presented in [5–7]. In [8], the general dynamic reflection coefficients of structures with BC that contain a mass, a spring and a damper were derived via the Laplace domain and used in active vibration suppression applications.

In [9], which is the most recent and comprehensive report on the subject at hand, modal analysis was applied to a fixed-damped structure. Explicit expressions for the eigenvalues and eigenfunctions were derived, and the solution was presented in an infinite series form. However, the series coefficients were not calculated and a question was raised whether that is possible at all. Without the series coefficients any practical use of the solution, such as calculation of the actual time response or finite dimension approximation by modal truncation, cannot be made. Preliminary results regarding the modal representation were given in [10] by using expansion of the infinite dimension transfer functions [8]. In this paper the complete modal series solution is developed in a direct manner. The general case of a structure with viscous damping at both boundaries is considered. This includes fixed-damped and free-damped structures as special cases by setting the damping coefficient to infinity and zero, respectively. Explicit expressions for all the solution components, including the coefficients, are found. The key for the solution derivation turns out to be the special orthogonality property of the damped eigenfunctions.

For systems governed by the wave equation it is natural to consider the free response in terms of propagating waves. The traveling wave solution of the fixed-damped structure is derived for the first wave reflection from each boundary in [9]. In [11], the traveling wave solution of a damped–damped structure was developed via the Laplace domain approach and in [12] by a direct time domain derivation. The response in [12] was represented by a single propagating wave that is constructed according to the wave reflections from the boundaries. In this paper it is shown how the modal series solution can be transformed to retrieve the traveling wave solution. The concept of "end waves" is identified here as a part of the modal series and its role in the continuity of the total displacement response is analyzed.

2. Problem statement and the solution via modal analysis

As an example of second order structures, we consider a uniform and homogeneous flexible rod of length L with viscous dampers D_1 and D_2 at both ends and no internal damping, as illustrated in Fig. 1.

The rod is governed by the well-known wave equation [1–4]:

$$\theta_{tt}(x,t) - c^2\theta_{xx}(x,t) = 0 \tag{1}$$

where $\theta(x,t)$ is the torsion angle of the rod at a distance x and time t, $c = (G/\rho)^{1/2}$ is the wave propagation velocity, G is the shear elasticity modulus and ρ is the material density. The BC are given by

$$I_pG\theta_x(x,t) = D_1\theta_t(x,t) \quad \text{at } x = 0, \qquad I_pG\theta_x(x,t) = -D_2\theta_t(x,t) \quad \text{at } x = L \tag{2a,b}$$

where I_p is the polar moment of inertia. It is assumed throughout the paper that at least one of the dampers is non-zero. For limit values of the dampers, Eq. (2) retrieves the basic geometrical BC: $D_i = 0$ for free ends and $D_i \to \infty$ for fixed. The rod is subjected to non-zero initial conditions (IC) of the displacement and the velocity:

$$\theta(x,0) = g(x), \qquad \theta_t(x,0) = v(x). \tag{3}$$

The function $g(x)$ has to be continuous but may be non-zero at the ends. Both $g(x)$ and $v(x)$ must satisfy the geometrical BC, which means that both of them are zero at a fixed end. A more detailed discussion about continuity requirements of the IC appears after the example in Section 3.1.

2.1. Separation of variables

The common method of solving partial differential equations in classical vibration theory is the separation of variables, [1–4]. We briefly present the basic steps of the method because those equations are needed later in the derivation. Substituting the assumed solution $\theta(x,t) = X(x)T(t)$ in (1) yields

$$X_{xx}(x) = \lambda^2 X(x), \qquad T_{tt}(t) = c^2\lambda^2 T(t) \tag{4a,b}$$