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Wave momentum in models of generalized continua

Martine Rousseau, Gérard A. Maugin *

Université Pierre et Marie Curie – Paris 6, Institut Jean Le Rond d'Alembert, UMR CNRS 7190, Case 162, 4 place Jussieu, 75252 Paris Cedex 05, France

A R T I C L E   I N F O

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Dedicated to (Volodya) Vladimir I. Alshits on the occasion of his 70th anniversary.

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A B S T R A C T

The possibility of associating the notion of quasi-particles with elastic wave modes is explored for three basic models of generalized continua: strain-gradient model (weak nonlocality), elasticity with a microstructure such as in Cosserat/micropolar materials, and a true nonlocal model involving the long-range interactions in the underlying crystal lattice. In each case a simplified one-dimensional model is considered. Approximate solutions involving small parameters and exhibiting scale effects are obtained for the Newtonian-like motion of associated quasi-particles. Interpretation for alternate wave-like and quasi-particle-like behaviors is given.

1. Introduction

In this contribution we combine two notions. One is that of a generalized continuum. One of us tried elsewhere to clearly define this notion together with its most frequent realizations [1]. These include three different paths. One is the introduction of a microstructure at each material point, in the simplest case, a rigidly rotating one just like in so-called Cosserat continua. This idea indeed goes back to the Cosserat brothers [2], with more modern presentations and full dynamics by, e.g., Eringen [3] and Nowacki [4]. The second realization is that which describes the local spatial displacement variation with a higher degree of accuracy than usual in the standard theory of elasticity. This is referred to as gradient theories, the second gradient of the displacement or first gradient of the strain being generally sufficient to arrive at some new effective improvement. This clearly is of interest wherever the strain is not spatially uniform. Such considerations go back to Le Roux [5,6] and in more recent times to Mindlin and Tiersten [7] and Toupin [8]. This is often referred to as a weakly nonlocal approach. The third path is that of strongly nonlocal theories of continua where the mechanical response at a material point depends on a larger spatial domain than the immediate neighborhood of the considered point. The idea may be traced back all the way to Duhem [9]. Syntheses on this type of approach are given by Kunin [10] and Eringen [11]. This is strongly correlated with a view to accounting for typical crystal properties (e.g., dispersion) while remaining in a continuum framework. The three paths of generalization are of interest in the study of elastic field singularities (at cracks, dislocations, etc.) as exemplified by the work of Lazar and Maugin [12] that compares the results in statics for the three schemes.

The second notion is that of wave momentum as initially understood by Brenig [13]. This is the momentum that is associated with a wave motion when written back in the reference configuration. Its nicest introduction is through the

* Corresponding author.
E-mail addresses: martine.rousseau@upmc.fr (M. Rousseau), gerard.maugin@upmc.fr (G.A. Maugin).

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writing down of canonical conservation laws that are associated with standard field equations by application of Noether's theorem of invariant theory. This viewpoint is expended in detail in a recent monograph [14]. The conservation law of interest in this context indeed is the conservation of wave momentum. We have shown elsewhere [15,16] how *quasi-particles* could be associated with wave motion in the considered standard (linear or physically nonlinear, weakly visco-, electro-, etc.) elastic medium. This was achieved essentially by integrating the *dynamic* equation of conservation of wave momentum over a volume representative of the wave motion.

The path for a combination of the two notions was paved in various papers: for Cosserat continua by Maugin [17] including all inertial dynamical effects, for gradient media by Maugin and Trimarco [18] also including inertia, and for nonlocal elasticity by Vukobrat and Kuzmanovic [19]. The first two cases are illustrated in Maugin [1] but the notion of total wave momentum and quasi-particle dynamics in these schemes of matter were not yet envisaged, which is dutifully examined here in simplified cases (e.g., motions depending only on one space variable and with one activated elastic displacement component only).

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The first case examined in Section 2 is that of a weakly nonlocal material according to a special model due to Aifantis [20] in which only one additional material coefficient (a characteristic length) is involved. This is considered for the sake of simplicity as it allows one to show the essential change brought to the wave momentum in a case where there exists physical dispersion.

Section 3 is devoted to the simple case of a Cosserat continuum in which one (transverse) elastic displacement couples with the internal rotation of a rigid microstructure. Accordingly, this is a two-degree of freedom system in which the essentially acoustic mode of propagation is altered by the coupling with this internal rotation. Both displacement and internal rotation participate in the definition of the wave momentum in agreement with the canonical definition of this quantity (all degrees of freedom contribute to the conservation of wave momentum; see Ref. [14], Chapter 4 on field theory).

Section 4 considers a one-dimensional case of (strong) nonlocal elasticity. In order to proceed far enough a typical nonlocality has to be assumed. Here, following Eringen [11,21–23], the nonlocality is chosen so as to reproduce crystal properties with the typical crystalline periodic structure (the celebrated Born–Kármán model; cf. Born and Huang [24]). As we shall see, this creates a specific situation where in the end we find that the notion of wave momentum is completely adapted to the known wave solution.

Note. Identical symbols can be used in the three approaches but they may refer to different entities without ambiguity.

Caveat. In this paper the quantities denoted by symbol M are in fact masses per unit area orthogonal to the direction of wave propagation. In previous works devoted to surface waves the same symbol corresponded to a mass per unit length in the direction orthogonal to the sagittal plane.

2. Weak nonlocality and gradient model of elasticity

2.1. Simplified problem

The relevant equations are recalled in Appendix A. However, we further simplify the problem by considering a propagation along axis $x_1 = x$, no dependence on the orthogonal coordinate x_2, and the only involved elastic displacement is a transverse one, $u_3 = u$ (had we considered a surface wave, u_3 would be a shear-horizontal (SH) component). With the above simplifications, Eq. (A.2) yields the only nonvanishing stress component

$$\sigma = \sigma_{31} = \mu\, u_{,x} + \alpha\mu\delta^2 u_{,xxx}, \tag{2.1}$$

where we set $c_{44} = \mu$ and introduced a numerical factor $\alpha = \pm 1$ which allows us to play with the sign of a possible dispersion (this matter is thoroughly discussed by Askes and Aifantis [25]).

2.2. Wave solution

Equation of motion (A.1) yields the dispersion relation

$$\omega^2 = c_0^2\, k^2\, \left(1 + \alpha\, \delta^2 k^2\right), \qquad c_0^2 = \mu/\rho_0, \qquad k_0 \equiv \omega/\, c_0. \tag{2.2}$$

If $\delta = 0$, we are in standard linear elasticity for which we introduce the notation $k_0 = \omega/c_0$ and $\lambda_0 = 2\pi/k_0$. The case $\delta \neq 0$, but obviously small or relatively small compared to a characteristic macroscopic length, requires a short discussion about the possible dispersion. Let λ be the wavelength of the solution. We can envisage two cases: (i) a resonance between the scales of λ and δ, i.e., $\lambda \approx \delta$ with $\lambda \ll \lambda_0$, or (ii) $\delta \ll \lambda \ll \lambda_0$. The case $\lambda < \delta$ is not meaningful. We denote by $c = \omega/k$ the phase velocity of the solution. With the above convention for α, we have the following possibilities:

$$c \cong c_0 \left(1 \pm \frac{1}{2}\delta^2 k^2\right), \qquad k \cong k_0 \left(1 \mp \frac{1}{2}\delta^2 k^2\right), \qquad \lambda \cong \lambda_0 \left(1 \pm \frac{1}{2}\delta^2 k^2\right). \tag{2.3}$$