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Analytic mode-matching for acoustic scattering in three dimensional waveguides with flexible walls: Application to a triangular duct

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ABSTRACT

An analytic mode-matching method suitable for the solution of problems involving scattering in three-dimensional waveguides with flexible walls is presented. Prerequisite to the development of such methods is knowledge of closed form analytic expressions for the natural fluid–structure coupled waveforms that propagate in each duct section and the corresponding orthogonality relations. In this article recent theory [J.B. Lawrie, Orthogonality relations for fluid–structural waves in a 3-D rectangular duct with flexible walls, Proc. R. Soc. A. 465 (2009) 2347–2367] is extended to construct the non-separable eigenfunctions for acoustic propagation in a three-dimensional rectangular duct with four flexible walls. For the special case in which the duct cross-section is square, the symmetrical nature of the eigenfunctions enables the eigenmodes for a right-angled, isosceles triangular duct with flexible hypotenuse to be deduced. The partial orthogonality relation together with other important properties of the triangular modes are discussed. A mode-matching solution to the scattering of a fluid–structure coupled wave at the junction of two identical semi-infinite ducts of triangular cross-section is demonstrated for two different sets of "junction" conditions.

1. Introduction

The scattering of waves in ducts or channels has long been of interest to scientists and engineers. Analytic mode-matching provides an appealing approach to the solution of many such problems. Traditionally the method has been restricted to canonical geometries in which the boundary value problems involve a governing equation such as Laplace's or Helmholtz's and in which the duct/channel walls are described by simple conditions (soft, hard or impedance). The underlying eigensystems for such boundary value problems are Sturm–Liouville in type and thus have well defined orthogonality properties. For more complicated geometries and/or ducts bounded by surfaces described by high-order conditions (such as the thin plate equation) alternative solution methods were necessary [1–3]. The past decade has seen a dramatic change in this situation. Hybrid mode-matching methods have been devised to deal with more complicated geometries [4–9] and the theory underpinning wave propagation in two-dimensional (2D) ducts with high order boundary conditions has been extensively developed [10,11].

Problems involving wave propagation in three dimensional (3D) ducts or channels with flexible walls remain, however, both challenging and of considerable interest to engineers [12–17]. Although mode-matching methods generally neglect the effects of break-out, they do enable physical insight into the underlying scattering processes and also provide benchmark solutions for fully numerical approaches. Thus, the development of a hybrid analytic–numerical mode-matching method

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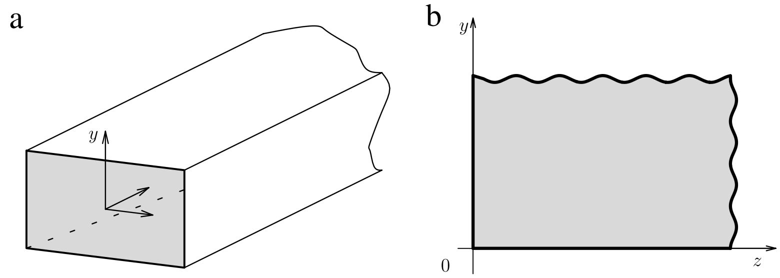

Fig. 1. (a) The rectangular 3D duct and (b) its yz-cross-section for $y \geq 0, z \geq 0$.

to address this class of problem would be a significant asset to the scientific community. Recent advances in this respect have been forged by Lawrie [18] who established much of the mathematical theory underlying acoustic propagation in a 3D rectangular duct with one flexible wall. A second article extends the theory to ducts with porous linings, internal structures or orthotropic boundaries, [19]. For each of the ducts considered in [18,19], the "corner conditions" applied along the length of the duct where the flexible wall meets the adjacent rigid wall dictate that the eigenmodes are non-separable in form. As a result only a "partial" orthogonality relation can be constructed and this, of course, complicates the mode-matching procedure. In contrast, Mondal et al. [20] consider a 3D problem involving the scattering of flexural gravity waves in a rectangular channel due to a crack in a floating ice sheet. In this case, as in [21], the nature of the "corner conditions" are such that the 3D eigenmodes are separable and, thus, the (generalised) orthogonality relation (OR) is more straightforward to apply.

The aim of this article is to develop a mode-matching approach for 3D ducts with flexible walls and in which the eigenfunctions are non-separable. In Section 2, the theory established in [18,19] is extended to a rectangular duct in which all four walls are flexible. Symmetry is then used to construct, in Section 3, the eigenmodes corresponding to acoustic propagation in a right angled, isosceles triangle in which the hypotenuse comprises a thin elastic plate whilst the other two walls are rigid. The (partial) orthogonality relation and other relevant properties of the eigenmodes are stated. In Section 4, a typical problem involving the scattering of an incident fluid–structural mode at the junction two identical semi-infinite ducts of triangular cross-section is considered. The solution to this problem is crucially dependent on the edge conditions applied at the junction of the two plates. It is demonstrated that an analytic mode-matching scheme can be constructed by which to determine the scattered field for two distinct sets of junction conditions. A comprehensive discussion of the method is presented in Section 5.

2. Propagation in a 3D duct with four flexible walls

A 3D duct of rectangular cross-section occupies the region $-\infty < \bar{x} < \infty, -\bar{a} \leq \bar{y} \leq \bar{a}, -\bar{b} \leq \bar{z} \leq \bar{b}$ where $(\bar{x}, \bar{y}, \bar{z})$ are dimensional Cartesian coordinates. The interior region of the duct contains a compressible fluid, of density ρ and sound speed c, whilst the region exterior to the duct is *in vacuo*. Each wall of the duct is flexible and thus able to move/vibrate in response to acoustic excitation. Under the assumption of harmonic time dependence, $e^{-i\omega\bar{t}}$, the fluid velocity potential is expressed in terms of the time independent potential by $\bar{\Phi}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \bar{\phi}(\bar{x}, \bar{y}, \bar{z})e^{-i\omega\bar{t}}$. It is convenient to non-dimensionalise the boundary value problem with respect to length and time scales k^{-1} and ω^{-1} respectively, where $\omega = ck$ and k is the fluid wavenumber. Thus, non-dimensional coordinates are defined by $x = k\bar{x}$ etc. Similarly, $\bar{\phi} = \omega\phi/k^2$ etc.

Due to coupling between the fluid and wall motions, the natural waves that travel within the duct are, in general, not separable with respect to y and z. It can, however, be assumed that they propagate in the positive x direction. The non-dimensional, time-independent velocity potential then assumes the form

$$\phi(x, y, z) = \sum_{n=0}^{\infty} B_n \Psi_n(y, z) e^{is_n x}, \quad x > 0 \tag{1}$$

where B_n is the amplitude of the nth travelling wave, s_n is the non-dimensional axial wavenumber (assumed to be either positive real or have positive imaginary part) and the non-separable eigenmodes $\Psi_n(y, z), n = 0, 1, 2, \ldots$ are to be determined.

The modes of propagation can be symmetric with respect to both y and z, anti-symmetric with respect to both y and z or a combination of symmetric in y (or z) and anti-symmetric in the other coordinate. For the sake of brevity, only fully symmetric modes will be discussed and the reader is referred to [19] where anti-symmetric modes for a duct of this class are discussed.

For modes that are symmetric with respect to both y and z, the eigensystem can be simplified. In this case, it is appropriate to consider wave propagation in a duct comprising two rigid walls, lying along $y = 0, 0 \leq z \leq b$, and $z = 0, 0 \leq y \leq a$ and two thin elastic plates lying along $y = a, 0 \leq z \leq b$, and $z = b, 0 \leq y \leq a$ (see Fig. 1(b)). It is convenient to treat the